A short guide to p-torsion of abelian varieties in characteristic p

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Abstract. There are many equivalent ways to describe the p-torsion of a principally polarized abelian variety in characteristic p. We briefly explain these methods and then illustrate them for abelian varieties A of arbitrary dimension g in several important cases, including when A has p-rank f and a-number 1 and when A has p-rank f and a-number $g-f$. We provide complete tables for abelian varieties of dimension at most four.

1. Introduction

One attribute of every complex abelian variety of dimension g is that its p-torsion points form a group of order p^{2g}. In contrast, the p-torsion points on a g-dimensional abelian variety defined over an algebraically closed field k of characteristic p form a group of order at most p^g. Exceptional research has emerged in response to this phenomenon, from early work on Picard schemes to recent results on stratifications of moduli spaces of abelian varieties.

The p-torsion of a principally polarized abelian variety defined over k can be described in terms of a group scheme or a Dieudonné module. It can be classified using its final type or Young type. It can be identified with an element in the Weyl group of the sympletic group or with a cycle class in the tautological ring of $\mathcal{A}_g$.

In this paper, we briefly summarize the main types of classification. We give a thorough description of the p-torsion of a principally polarized abelian variety A of arbitrary dimension g in several important cases, including when A has p-rank f and a-number 1, and when A has p-rank f and a-number $g-f$. We provide complete tables for the p-torsion types that occur for $g \leq 4$, including the sixteen types of p-torsion that occur for abelian varieties of dimension four.

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2. Methods to classify the p-torsion

Let k be an algebraically closed field of characteristic p. Let $\mathcal{A}_g := \mathcal{A}_g \otimes k$ be the moduli space of principally polarized abelian varieties of dimension g defined over k. Let $A \in \mathcal{A}_g(k)$ be an abelian variety of dimension g defined over k.

Consider the multiplication-by-p morphism $[p] : A \to A$ which is a proper flat morphism of degree p^{2g}. It factors as $[p] = \mathrm{Ver} \circ \mathrm{Fr}$. Here $\mathrm{Fr} : A \to A^{(p)}$ is the relative Frobenius morphism coming from the p-power map on the structure sheaf; it is purely inseparable of degree p^g. The Verschiebung morphism $\mathrm{Ver} : A^{(p)} \to A$ is the dual of Fr. If A is principally polarized, then $\mathrm{im}(\mathrm{Fr}) = \ker(\mathrm{Ver})$ and $\mathrm{im}(\mathrm{Ver}) = \ker(\mathrm{Fr})$.

The kernel of $[p]$ is $A[p]$, the p-torsion of A. We summarize several different ways of describing $A[p]$.

2.1. Group schemes. The p-torsion $A[p]$ is a finite commutative group scheme annihilated by p with rank p^{2g}, again having morphisms Fr and Ver. Then $A[p]$ is called a quasi-polarized BT_1 k-group scheme (short for quasi-polarized truncated Barsotti-Tate group of level 1). The quasi-polarization implies that $A[p]$ is symmetric. These group schemes were classified independently by Kraft (unpublished) [**Kra**] and by Oort [**Oor01**]. A complete description of this topic can be found in [**Oor01**] or [**Moo01**].

Example 2.1. Consider the constant group scheme $\mathbb{Z}/p = \mathrm{Spec}(\oplus_{\gamma \in \mathbb{Z}/p} k)$ with co-multiplication $m^*(\gamma) = \sum_{\delta \in \mathbb{Z}/p} \gamma\delta \otimes \delta^{-1}$ and co-inverse $\mathrm{inv}^*(\gamma) = \gamma^{-1}$. Also consider μ_p which is the kernel of Frobenius on $\mathbb{G}_m$. As a k-scheme, $\mu_p \simeq \mathrm{Spec}(k[x]/(x^p - 1))$ with co-multiplication $m^*(x) = x \otimes x$ and co-inverse $\mathrm{inv}^*(x) = x^{-1}$. If E is an ordinary elliptic curve then $E[p] \simeq \mathbb{Z}/p \oplus \mu_p$. We denote this group scheme by L.

Example 2.2. Let α_p be the kernel of Frobenius on $\mathbb{G}_a$. As a k-scheme, $\alpha_p \simeq \mathrm{Spec}(k[x]/x^p)$ with co-multiplication $m^*(x) = x \otimes 1 + 1 \otimes x$ and co-inverse $\mathrm{inv}^*(x) = -x$. The isomorphism type of the p-torsion of any two supersingular elliptic curves is the same. If E is a supersingular elliptic curve, we denote the isomorphism type of its p-torsion by $I_{1,1}$. By [**Gor02**, Ex. A.3.14], $I_{1,1}$ fits into a non-split exact sequence of the form $0 \to \alpha_p \to I_{1,1} \to \alpha_p \to 0$. The image of the embedded α_p is unique and is the kernel of both Frobenius and Verschiebung.

Example 2.3. Let A be a supersingular non-superspecial abelian surface. In other words, A is isogenous, but not isomorphic, to the direct sum of two supersingular elliptic curves. Let $I_{2,1}$ denote the isomorphism class of the group scheme $A[p]$. By [**Gor02**, Ex. A.3.15], there is a filtration $H_1 \subset H_2 \subset I_{2,1}$ where $H_1 \simeq \alpha_p$, $H_2/H_1 \simeq \alpha_p \oplus \alpha_p$, and $I_{2,1}/H_2 \simeq \alpha_p$. Also H_2 contains both the kernel G_1 of Frobenius and the kernel G_2 of Verschiebung. There is an exact sequence $0 \to H_1 \to G_1 \oplus G_2 \to H_2 \to 0$.

The p-rank and a-number. Two invariants of (the p-torsion of) an abelian variety are the p-rank and a-number. The *p-rank* of A is $f = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, A[p])$. Then p^f is the cardinality of $A[p](k)$. The *a-number* of A is $a = \dim_k \mathrm{Hom}(\alpha_p, A[p])$. It is well-known that $0 \leq f \leq g$ and $0 \leq a + f \leq g$.

In Example 2.1, $f = 1$ and $a = 0$. In Example 2.2, $f = 0$ and $a = 1$. The group scheme $I_{2,1}$ in Example 2.3 has p-rank 0 since it is an iterated extension of copies of α_p and has a-number 1 since $\ker(V^2) = G_1 \oplus G_2$ has rank p^3.

An abelian variety A of dimension g is *ordinary* if $A[p]$ has p-rank $f = g$. If A is ordinary then $A[p] \simeq L^g$. At the other extreme, A is *superspecial* if $A[p]$ has a-number $a = g$. In this case, $A \simeq E^g$ for a supersingular elliptic curve E and $A[p] \simeq I_{1,1}^g$ [**LO98**, 1.6].

2.2. Covariant Dieudonné modules. One can describe the p-torsion $A[p]$ using the theory of covariant Dieudonné modules. This is the dual of the contravariant theory found in [**Dem86**]; see also [**Gor02**, A.5]. Briefly, let σ denote the Frobenius automorphism of k. Consider the non-commutative ring $\mathbb{E} = k[F, V]$ generated by semi-linear operators F and V with the relations $FV = VF = 0$ and $F\lambda = \lambda^\sigma F$ and $\lambda V = V\lambda^\sigma$ for all $\lambda \in k$. Let $\mathbb{E}(A, B)$ denote the left ideal $\mathbb{E}A + \mathbb{E}B$ of $\mathbb{E}$ generated by A and B. A deep result is that the Dieudonné functor D gives an equivalence of categories between BT_1 group schemes $\mathbb{G}$ (with rank p^{2g}) and finite left $\mathbb{E}$-modules $D(\mathbb{G})$ (having dimension $2g$ as a k-vector space). If $\mathbb{G}$ is quasi-polarized, then there is a sympletic form on $D(\mathbb{G})$.

Here are some examples of Dieudonné modules: $D(\mathbb{Z}/p \oplus \mu_p) \simeq \mathbb{E}/\mathbb{E}(F, 1 - V) \oplus \mathbb{E}/\mathbb{E}(V, 1 - F)$, [**Gor02**, Ex. A.5.1 & 5.3]. In Lemma 3.1, we will show that $D(I_{1,1}) \simeq \mathbb{E}/\mathbb{E}(F + V)$ and that $D(I_{2,1}) \simeq \mathbb{E}/\mathbb{E}(F^2 + V^2)$.

The p-rank of $A[p]$ is the dimension of $V^g D(\mathbb{G})$. The a-number of $A[p]$ equals $g - \dim(V^2 D(\mathbb{G}))$ [**LO98**, 5.2.8].

2.3. Final types. The isomorphism type of a symmetric BT_1 group scheme $\mathbb{G}$ over k can be encapsulated into combinatorial data. This topic can be found in [**Oor01**]. If $\mathbb{G}$ has rank p^{2g}, then there is a *final filtration* $N_1 \subset N_2 \subset \cdots \subset N_{2g}$ of $D(\mathbb{G})$ as a k-vector space which is stable under the action of V and F^{-1} so that $i = \dim(N_i)$. In particular, $N_g = VD(\mathbb{G})$. If $\mathbb{G}$ is quasi-polarized, then N_{2g-i} and N_i are orthogonal under the symplectic pairing.

The *final type* of $\mathbb{G}$ is $\nu = [\nu_1, \ldots, \nu_r]$ where $\nu_i = \dim(V(N_i))$. The final type of $\mathbb{G}$ is canonical, even if the final filtration is not. There is a restriction $\nu_i \leq \nu_{i+1} \leq \nu_i + 1$ on the final type. All sequences satisfying this restriction occur. This implies that there are 2^g isomorphism types of symmetric BT_1 group schemes of rank p^{2g}. The p-rank is $\max\{i \mid \nu_i = i\}$ and the a-number is $g - \nu_g$.

For example, suppose $\mathbb{G}$ is ordinary. Then V acts invertibly on $VD(L)$ and so $\nu_g = g$, which implies $\nu_i = i$ for $1 \leq i \leq g$. Thus $\nu = [1, 2, \ldots g]$ when $\mathbb{G}$ is ordinary. At the other extreme, when $\mathbb{G}$ is superspecial then $\dim(V^2(D(\mathbb{G})) = 0$ and so $\nu = [0, 0, \ldots, 0]$.

Together with Ekedahl, Oort used the classification by final type to stratify $\mathcal{A}_g$. The stratum of $\mathcal{A}_g$ whose points have final type ν is locally closed and quasi-affine with dimension $\sum_{i=1}^g \nu_i$ [**Oor01**, Thm. 1.2].

2.4. Young types. Another combinatorial method to describe the isomorphism type of $\mathbb{G}$ uses a Young diagram. Given a final type ν, for $1 \leq j \leq g$, let $\mu_j = \#\{i \mid 1 \leq i \leq g \mid i - \nu_i \geq j\}$. The sequence μ_j is strictly decreasing. Consider the Young diagram with μ_j squares in the jth row. The *Young type* of $\mathbb{G}$ is $\mu = \{\mu_1, \mu_2, \ldots\}$, where one eliminates all μ_j which equal 0. The p-rank is $g - \mu_1$ because this equals $\#\{i \mid i - \nu_i = 0\}$. The a-number is $a = \max\{j \mid \mu_j \neq 0\}$ because this equals $g - \nu_g$.

For example, in the ordinary case, $i - \nu_i = 0$ when $1 \leq i \leq g$ and so $\mu = \emptyset$. In the superspecial case, $\nu_i = 0$ and $\mu_j = \#\{i \mid j \leq i \leq g\} = g - j + 1$ and thus $\mu = \{g, g - 1, \ldots, 1\}$.

The Young type of $\mathbb{G}$ was introduced by Van der Geer [**vdG99**] as a means of describing the Ekedahl-Oort strata in terms of degeneration loci for maps between flag varieties. The codimension in $\mathcal{A}_g$ of the stratum whose points have Young type μ is $\sum_{j=1}^{a} \mu_j$.

2.5. Elements of the Weyl group. One can associate to μ an element ω of the Weyl group W_g of the sympletic group Sp_{2g} [**vdG99**]. Here W_g is identified with the subgroup of all $\omega \in S_{2g}$ such that $\omega(i)+\omega(2g+1-i) = 2g+1$ for $1 \leq i \leq g$. This subgroup is generated by the following involutions: $s_i = (i, i+1)(2g-i, 2g+1-i)$ for $1 \leq i < g$; and $s_g = (g, g+1)$.

Given a Young type μ, one defines ω as follows. For $1 \leq i \leq g$, let $\omega(i) = c$ (respectively $\omega(i) = g + c$) if i is the cth number such that $\mu_i = \mu_{i+1}$ (respectively $\mu_i \neq \mu_{i+1}$). For $1 \leq i \leq g$, let $\omega(2g+1-i) = 2g+1-\omega(i)$. This yields an element of W_g. One can express ω as a word in the involutions $s_1, \ldots, s_g$ of S_{2g}, although this expression is not unique.

For example, in the ordinary case where $\mu = \emptyset$, then ω is given by $\langle 1, \ldots, 2g\rangle \xrightarrow{\omega} \langle g+1, \ldots, 2g, 1, \ldots, g\rangle$. In the superspecial case where $\mu = \{g, \ldots, 1\}$, then $\omega = \mathrm{id}$. Further examples with $g \leq 4$ are in Section 4.

We briefly explain the importance of the Weyl group characterization. There is a second filtration of $D(\mathbb{G})$ which is stable under the action of F and V^{-1}, which we denote by $N_1' \subset N_2' \subset \cdots \subset N_{2g}'$. Then ω measures the interaction between these two filtrations.

For example, when $\mathbb{G}$ is ordinary ($f = g$) then $N_g \cap N_g' = 0$. Informally speaking, this means that the intersection of $\mathrm{Im}(V)$ and (a twist under σ of) $\mathrm{Im}(F)$ is trivial. When $\mathbb{G}$ is superspecial ($a = g$), then $\dim(N_i \cap N_g') = i$ for $1 \leq i \leq g$. Informally speaking, this implies that N_i is contained in (a twist under σ of) $\mathrm{Im}(F)$. In general, $\dim(N_i \cap N_g') \geq i - \nu_i$. The a-number is $\dim(VD(\mathbb{G}) \cap FD(\mathbb{G})) = \dim(N_g \cap N_g')$.

One can calculate the cycle classes of the closures of the Ekedahl-Oort strata in the tautological ring of $\mathcal{A}_g$. Let λ_i for $1 \leq i \leq g$ be the Chern classes of the Hodge bundle of $\mathcal{A}_g$. These classes generate the tautological subring of $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ and satisfy $(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g) = 1$ [**vdG99**, Thm. 1.1].

3. Important examples

3.1. Abelian varieties with p-rank f. Given g and f such that $0 \leq f \leq g$, let $V_{g,f}$ denote the stratum of $\mathcal{A}_g$ whose points correspond to principally polarized abelian varieties A of dimension g with $f_A \leq f$. The generic point of $V_{g,g} = \mathcal{A}_g$ corresponds to an abelian variety with p-rank g, a-number 0, and $A[p] \cong L^g$. If $f < g$, every component of $V_{g,f}$ has codimension $g - f$, and its generic point has a-number 1, [**NO80**, Thm. 4.1]. In this section, we describe the p-torsion group scheme that occurs for the generic point(s) of $V_{g,f}$.

LEMMA 3.1. *Let $r \in \mathbb{N}$. There is a unique symmetric BT_1 group scheme of rank p^{2r} with p-rank 0 and a-number 1, which we denote $I_{r,1}$. The covariant Dieudonné module of $I_{r,1}$ is $\mathbb{E}/\mathbb{E}(F^r + V^r)$.*

PROOF. Let $I_{r,1}$ be a symmetric BT_1 group scheme of rank p^{2r} with p-rank 0 and a-number 1. It is sufficient to show that the final type of $I_{r,1}$ is uniquely determined. The p-rank 0 condition implies that V acts nilpotently on $D(I_{r,1})$, so $\nu_1 = 0$. The a-number 1 condition implies that $r - 1$ is the dimension of $V^2D(I_{r,1})$, so $\nu_r = r-1$. The restrictions on ν_i imply that there is a unique final type possible for $I_{r,1}$, namely $\nu = [0, 1, \ldots, r-1]$.

Consider $D = \mathbb{E}/\mathbb{E}(F^r + V^r)$. Note that $F^{r+1} = 0$ and $V^{r+1} = 0$ on D. Then D is an $\mathbb{E}$-module with dimension $2r$ as a k-vector space. It has basis $\{F, \ldots, F^r, 1, V, \ldots, V^{r-1}\}$. Then VD has basis $\{V, \ldots, V^{r-1}, F^r\}$ and V^2D has basis $\{V^2, \ldots, V^{r-1}, F^r\}$. Thus D has a-number 1. Continuing, one sees that V is nilpotent on D and thus the p-rank of D is 0. Thus D must be the covariant Dieudonné module $D(I_{r,1})$. □

PROPOSITION 3.2. *Let $A \in \mathcal{A}_g(k)$ be a principally polarized abelian variety of dimension g with p-rank f and a-number 1. Then $A[p] \simeq L^f \oplus I_{g-f,1}$. The covariant Dieudonné module of $A[p]$ is*

$$D \simeq (\mathbb{E}/\mathbb{E}(F, 1-V) \oplus \mathbb{E}/\mathbb{E}(V, 1-F))^f \oplus \mathbb{E}/\mathbb{E}(F^{g-f} - V^{g-f}).$$

The final type of $A[p]$ is $\nu = [1, \ldots, f, f, \ldots, g-1]$ (or $[0, \ldots, 0]$ if $f = 0$). The Young type is $\mu = \{g-f\}$ (or $\emptyset$ if $f = g$).

PROOF. The group scheme $A[p]$ must include f copies of L along with a group scheme of rank $p^{2(g-f)}$ with p-rank 0 and a-number 1. By Lemma 3.1, the only possibility for the latter is $I_{g-f,1}$. The statement about the Dieudonné module follows immediately. For the final type, note that $\nu_g = g-1$ since $A[p]$ has a-number 1 and $\nu_f = f$ since $A[p]$ has p-rank f. The numerical restrictions on ν_i then imply that $\nu = [1, \ldots, f, f, \ldots, g-1]$. The Young type follows by direct calculation. □

If $f < g$, one can show that the group scheme $L^f \oplus I_{g-f,1}$ corresponds to the element ω of the Weyl group such that $\omega(f+1) = 1$ and $\omega : \{1, \ldots, g\} - \{f+1\} \mapsto \{g+1, \ldots, 2g-1\}$ is increasing. The cycle class of the (reduced) stratum $V_{g,f}$ in the tautological ring of $\mathcal{A}_g$ is given by $(p-1)(p^2-1)\ldots(p^{g-f}-1)\lambda_{g-f}$ [**vdG99**, Thm. 2.4].

3.2. Abelian varieties with given a-number. Given g and f such that $0 \le f \le g$, let $T_{g,a}$ denote the stratum of $\mathcal{A}_g$ whose points correspond to principally polarized abelian varieties of dimension g with $a_A \ge a$. Then $T_{g,a}$ is irreducible unless $a = g$ [**vdG99**, Thm. 2.11]. In this section, we describe the p-torsion that occurs for the generic point(s) of $T_{g,a}$. It is known that $T_{g,a}$ has codimension $a(a+1)/2$. The generic point(s) of $T_{g,a}$ have a-number a and p-rank $g-a$.

PROPOSITION 3.3. *Let $A \in \mathcal{A}_g(k)$ be an abelian variety of dimension g with p-rank f and a-number $g-f$. Then $A[p] \simeq L^f \oplus (I_{1,1})^{g-f}$. The covariant Dieudonné module of $A[p]$ is*

$$D \simeq (\mathbb{E}/\mathbb{E}(F, 1-V) \oplus \mathbb{E}/\mathbb{E}(V, 1-F))^f \oplus (\mathbb{E}/\mathbb{E}(F+V))^{g-f}.$$

The final type is $\nu = [1, \ldots, f, \ldots, f]$ (or $[0, \ldots, 0]$ if $f = 0$). The Young type is $\mu = \{g-f, g-f-1, \ldots, 1\}$ (or $\emptyset$ if $g = f$).

PROOF. The group scheme $A[p]$ must include f copies of L along with a group scheme of rank $p^{2(g-f)}$ with p-rank 0 and a-number $g-f$. The only possibility for the latter is $g-f$ copies of $D(I_{1,1})$. The statement about the Dieudonné module follows immediately. For the final type, note that $\nu_g = f$ since $A[p]$ has a-number $g-f$ and $\nu_f = f$ since $A[p]$ has p-rank f. The numerical restrictions on ν_i then imply that $\nu = [1, \ldots, f, \ldots, f]$. The Young type follows by direct calculation. □

If $f > 0$, one can show that the group scheme $L^f \oplus (I_{1,1})^{g-f}$ corresponds to the element ω of the Weyl group $\langle 1, \dots, 2g\rangle \stackrel{\omega}{\to} \langle g+1, \dots, g+f, 1, \dots a, g+f+1, \dots 2g, a+1, \dots, g\rangle$. In [**vdG99**, Thm. 2.6], one finds a result on the cycle class of the (reduced) stratum $T_{g,a}$ in the tautological ring of $\mathcal{A}_g$.

3.3. Indecomposable group schemes. Almost all of the group schemes occuring in dimension $g \leq 4$ arise as direct sums of the examples from previous sections. To finish the tables in Section 4, we need to describe a few other group schemes, which are indecomposable.

A symmetric BT_1 group scheme $\mathbb{G}$ is *decomposable* if $\mathbb{G} \simeq \mathbb{G}_1 \oplus \mathbb{G}_2$ where $\mathbb{G}_1$ and $\mathbb{G}_2$ are nontrivial symmetric BT_1 group schemes; otherwise $\mathbb{G}$ is *indecomposable.* For example, the group schemes $I_{r,1}$ are indecomposable.

Some questions about a symmetric BT_1 group scheme $\mathbb{G}$ can be reduced to the case that $\mathbb{G}$ is indecomposable. We note that $A[p]$ can be decomposable even when A is a simple abelian variety. Here are some more examples of indecomposable group schemes.

An indecomposable group scheme of rank p^{2r} with p-rank 0 and $a = 2$.

LEMMA 3.4. *Let $r \in \mathbb{N}$ with $r \geq 3$. Let $D = \mathbb{E}/\mathbb{E}(F^{r-1} - V) \oplus \mathbb{E}/\mathbb{E}(V^{r-1} - F)$. Then D is the covariant Dieudonné module of an indecomposable symmetric BT_1 group scheme with rank p^{2r}, p-rank 0 and a-number 2, which we denote by $I_{r,2}$. It has final type $[0, 1, \dots, r-3, r-2, r-2]$ and Young type $\{r, 1\}$.*

PROOF. The given decomposition of D is the only possible decomposition of D into covariant Dieudonné modules, but neither of the factors in this decomposition is symmetric. Thus $I_{r,2}$ is indecomposable as a symmetric BT_1 group scheme.

Note that $F^r = 0$ (resp. $V^r = 0$) on the first (resp. second) factor of D. Then $D = N_{2r} = \langle 1, F, \dots, F^{r-1}\rangle \oplus \langle 1, V, \dots, V^{r-1}\rangle$. Thus D has dimension $2r$ as a k-vector space and $I_{r,2}$ has rank p^{2r}. Then $VD = \langle F^{r-1}\rangle \oplus \langle V, \dots, V^{r-1}\rangle = N_r$. Also $V^2D = 0 \oplus \langle V^2, \dots, V^{r-1}\rangle = N_{r-2}$. Thus D has a-number 2. Continuing, one sees that $\nu_i = i - 1$ for $1 \leq i \leq r-2$. In particular, V is nilpotent on D and thus the p-rank of D is 0.

More information on the final filtration is necessary to determine the final type of $I_{r,2}$. First, $F^{-1}(N_r) = \langle F^{r-2}, F^{r-1}\rangle \oplus \langle 1, V, \dots, V^{r-1}\rangle = N_{r+2}$. Second, $F^{-r+2}(N_r) = \langle F, \dots, F^{r-1}\rangle \oplus \langle 1, V, \dots, V^{r-1}\rangle = N_{2r-1}$. Then $VN_{2r-1} = 0 \oplus \langle V, \dots, V^{r-1}\rangle = N_{r-1}$ and $VN_{r-1} = \langle V^2, \dots, V^{r-1}\rangle = N_{r-2}$. Thus $\nu_{r-1} = r-2$. Then $I_{r,2}$ has final type $[0, 1, \dots, r-3, r-2, r-2]$ and Young type $\{r, 1\}$. □

LEMMA 3.5. *When $g = 3$ or $g = 4$, there is a unique indecomposable symmetric BT_1 group scheme $I_{g,2}$ of rank p^{2g} with p-rank 0 and a-number 2.*

PROOF. Let $\mathbb{G}$ be a symmetric BT_1 group scheme with rank p^{2g}, p-rank 0 and a-number 2. Its Young type is $\{g, i\}$ for some $i \in \{1, \dots, g-1\}$. There are exactly $\lfloor g/2 \rfloor$ such group schemes which are decomposable, namely $I_{r,1} \oplus I_{g-r,1}$ for $1 \leq r \leq g/2$. Thus there is a unique such $\mathbb{G}$ which is indecomposable when $g = 3$ of $g = 4$. By Lemma 3.4, it is $I_{g,2}$. □

One more indecomposable group scheme of dimension four. There is one more indecomposable group scheme which occurs for dimension $g \leq 4$, which we denote by $I_{4,3}$. It has covariant Dieudonné module $D(I_{4,3}) = \mathbb{E}/\mathbb{E}(F^2 - V) \oplus \mathbb{E}/\mathbb{E}(F - V) \oplus \mathbb{E}/\mathbb{E}(V^2 - F)$. Then $D(I_{4,3})$ has basis $\langle 1, F, F^2\rangle \oplus \langle 1, F\rangle \oplus \langle 1, V, V^2\rangle$.

One can check that V^2D has basis $0 \oplus 0 \oplus \langle V^2 \rangle$ and thus $I_{4,3}$ has a-number 3. Also $I_{4,3}$ has p-rank 0 since V acts nilpotently on $D(I_{4,3})$. By the process of elimination, $I_{4,3}$ has final type $[0, 0, 1, 1]$ and Young type $\{4, 3, 1\}$.

4. Complete tables for dimension at most four

For convenience, we provide tables for dimension $g \leq 4$. Some parts of these tables can be found in [**EvdG**]. The second column gives the codimension of the strata in $\mathcal{A}_g$.

4.1. The case $g = 1$:

Name	codim	f	a	ν	μ	ω	cycle class (reduced)
L	0	1	0	$[1]$	$\emptyset$	s_1	λ_0
$I_{1,1}$	1	0	1	$[0]$	$\{1\}$	1	$(p-1)\lambda_1$

4.2. The case $g = 2$:

Name	codim	f	a	ν	μ	ω	cycle class (reduced)
L^2	0	2	0	$[1, 2]$	$\emptyset$	$s_2s_1s_2$	λ_0
$L \oplus I_{1,1}$	1	1	1	$[1, 1]$	$\{1\}$	s_1s_2	$(p-1)\lambda_1$
$I_{2,1}$	2	0	1	$[0, 1]$	$\{2\}$	s_2	$(p-1)(p^2-1)\lambda_2$
$I_{1,1}^2$	3	0	2	$[0, 0]$	$\{2, 1\}$	1	$(p-1)(p^2+1)\lambda_1\lambda_2$

This is the smallest dimension for which the Newton polygon of A does not determine the group scheme $A[p]$. The Newton polygon $2G_{1,1}$ (supersingular, with four slopes of $1/2$) occurs for both $(I_{1,1})^2$ and $I_{2,1}$.

4.3. The case $g = 3$:

Name	codim	f	a	ν	μ	ω
L^3	0	3	0	$[1, 2, 3]$	$\emptyset$	$s_3s_2s_3s_1s_2s_3$
$L^2 \oplus I_{1,1}$	1	2	1	$[1, 2, 2]$	$\{1\}$	$s_2s_3s_1s_2s_3$
$L \oplus I_{2,1}$	2	1	1	$[1, 1, 2]$	$\{2\}$	$s_3s_1s_2s_3$
$L \oplus I_{1,1}^2$	3	1	2	$[1, 1, 1]$	$\{2, 1\}$	$s_1s_2s_3$
$I_{3,1}$	3	0	1	$[0, 1, 2]$	$\{3\}$	$s_3s_2s_3$
$I_{3,2}$	4	0	2	$[0, 1, 1]$	$\{3, 1\}$	s_2s_3
$I_{1,1} \oplus I_{2,1}$	5	0	2	$[0, 0, 1]$	$\{3, 2\}$	s_3
$I_{1,1}^3$	6	0	3	$[0, 0, 0]$	$\{3, 2, 1\}$	1

This is the smallest dimension for which the group scheme $A[p]$ does not determine the Newton polygon of A. If $A[p] \simeq I_{3,1}$, then the Newton polygon of A is usually $G_{1,2} + G_{2,1}$ (three slopes of $1/3$ and of $2/3$) but by [**Oor91**, Thm. 5.12] it can also be $3G_{1,1}$ (supersingular, with six slopes of $1/2$). The cycle classes for this table can be found in [**EvdG**, 15.2].

4.4. The case $g = 4$:

Name	codim	f	a	ν	μ	ω
L^4	0	4	0	$[1,2,3,4]$	$\emptyset$	$s_4s_3s_4s_2s_3s_4s_1s_2s_3s_4$
$L^3 \oplus I_{1,1}$	1	3	1	$[1,2,3,3]$	$\{1\}$	$s_3s_4s_2s_3s_4s_1s_2s_3s_4$
$L^2 \oplus I_{2,1}$	2	2	1	$[1,2,2,3]$	$\{2\}$	$s_4s_2s_3s_4s_1s_2s_3s_4$
$L^2 \oplus I_{1,1}^2$	3	2	2	$[1,2,2,2]$	$\{2,1\}$	$s_2s_3s_4s_1s_2s_3s_4$
$L \oplus I_{3,1}$	3	1	1	$[1,1,2,3]$	$\{3\}$	$s_4s_3s_4s_1s_2s_3s_4$
$L \oplus I_{3,2}$	4	1	2	$[1,1,2,2]$	$\{3,1\}$	$s_3s_4s_1s_2s_3s_4$
$I_{4,1}$	4	0	1	$[0,1,2,3]$	$\{4\}$	$s_4s_3s_4s_2s_3s_4$
$L \oplus I_{1,1} \oplus I_{2,1}$	5	1	2	$[1,1,1,2]$	$\{3,2\}$	$s_4s_1s_2s_3s_4$
$I_{4,2}$	5	0	2	$[0,1,2,2]$	$\{4,1\}$	$s_3s_4s_2s_3s_4$
$L \oplus I_{1,1}^3$	6	1	3	$[1,1,1,1]$	$\{3,2,1\}$	$s_1s_2s_3s_4$
$I_{1,1} \oplus I_{3,1}$	6	0	2	$[0,1,1,2]$	$\{4,2\}$	$s_4s_2s_3s_4$
$I_{1,1} \oplus I_{3,2}$	7	0	3	$[0,1,1,1]$	$\{4,2,1\}$	$s_2s_3s_4$
$I_{2,1} \oplus I_{2,1}$	7	0	2	$[0,0,1,2]$	$\{4,3\}$	$s_4s_3s_4$
$I_{4,3}$	8	0	3	$[0,0,1,1]$	$\{4,3,1\}$	s_3s_4
$I_{1,1}^2 \oplus I_{2,1}$	9	0	3	$[0,0,0,1]$	$\{4,3,2\}$	s_4
$I_{1,1}^4$	10	0	4	$[0,0,0,0]$	$\{4,3,2,1\}$	1

The cycle classes for this table can be found in [**EvdG**, 15.3].

It is not straight-forward to determine which Ekedahl-Oort strata lie in the boundary of which others. When $g = 4$, the answer to this question is given by the natural partial ordering on the Young type, which matches the Bruhat-Chevalley order on the elements of the Weyl group.

$$\begin{array}{ccccccccccccccc}
\{\emptyset\} & \to & \{1\} & \to & \{2\} & \to & \{3\} & \to & \{4\} & & & & & & \\
& & & & \downarrow & & \downarrow & & \downarrow & & & & & & \\
& & & & \{2,1\} & \to & \{3,1\} & \to & \{4,1\} & & & & & & \\
& & & & & & \downarrow & & \downarrow & & & & & & \\
& & & & & & \{3,2\} & \to & \{4,2\} & \to & \{4,3\} & & & & \\
& & & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
& & & & & & \{3,2,1\} & \to & \{4,2,1\} & \to & \{4,3,1\} & \to & \{4,3,2\} & \to & \{4,3,2,1\}
\end{array}$$

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