CURVES OF GIVEN p-RANK WITH TRIVIAL AUTOMORPHISM GROUP

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ABSTRACT. Let k be an algebraically closed field of characteristic $p > 0$. Suppose $g \geq 3$ and $0 \leq f \leq g$. We prove there is a smooth projective k-curve of genus g and p-rank f with no non-trivial automorphisms. In addition, we prove there is a smooth projective hyperelliptic k-curve of genus g and p-rank f whose only non-trivial automorphism is the hyperelliptic involution. The proof involves computations about the dimension of the moduli space of (hyperelliptic) k-curves of genus g and p-rank f with extra automorphisms.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic $p > 0$. If $g \geq 3$, there exist a k-curve C of genus g with $\mathrm{Aut}(C) = \{1\}$ and a hyperelliptic k-curve D of genus g with $\mathrm{Aut}(D) \simeq \mathbb{Z}/2$ (see, e.g., [16] and [8], respectively). In this paper, we extend these results to curves with given genus and p-rank.

If C is a smooth projective k-curve of genus g with Jacobian $\mathrm{Jac}(C)$, the *p-rank* of C is the integer f_C such that the cardinality of $\mathrm{Jac}(C)[p](k)$ is p^{f_C}. It is known that $0 \leq f_C \leq g$. We prove the following:

Theorem 1.1. *Suppose $g \geq 3$ and $0 \leq f \leq g$.*

(i) There exists a smooth projective k-curve C of genus g and p-rank f with $\mathrm{Aut}(C) = \{1\}$.

(ii) There exists a smooth projective hyperelliptic k-curve D of genus g and p-rank f with $\mathrm{Aut}(D) \simeq \mathbb{Z}/2$.

More generally, we consider the moduli space $\mathcal{M}_g$ of curves of genus g over k. The p-rank induces a stratification $\mathcal{M}_{g,f}$ of $\mathcal{M}_g$ so that the geometric points of $\mathcal{M}_{g,f}$ parametrize k-curves of genus g and p-rank at most f. Similarly, we consider the p-rank stratification $\mathcal{H}_{g,f}$ of the moduli space $\mathcal{H}_g$ of hyperelliptic k-curves of genus g. Our main results (Theorems 2.3 and 3.7) state that, for every geometric generic point η of $\mathcal{M}_{g,f}$ (resp. $\mathcal{H}_{g,f}$), the corresponding curve $\mathcal{C}_\eta$ satisfies $\mathrm{Aut}(\mathcal{C}_\eta) = \{1\}$ (resp. $\mathrm{Aut}(\mathcal{D}_\eta) \simeq \mathbb{Z}/2$).

2000 *Mathematics Subject Classification.* 11G20, 14H05.

Key words and phrases. p-rank, automorphism, curve, moduli, hyperelliptic.

* The third author was partially supported by NSF grant DMS-07-01303.

For the proof of the first result, we consider the locus $\mathcal{M}_g^\ell$ of $\mathcal{M}_g$ parametrizing k-curves of genus g which have an automorphism of order ℓ. Results from [7] and [16] allow us to compare the dimensions of $\mathcal{M}_{g,f}$ and $\mathcal{M}_g^\ell$. The most difficult case, when $\ell = p$, involves wildly ramified covers and deformation results from [2]. For the proof of the second result, we compare the dimensions of $\mathcal{H}_{g,f}$ and $\mathcal{H}_g^\ell$ using [9] and [10]. When $p = 2$, this relies on [17]. The hardest case for hyperelliptic curves is when $p \geq 3$, $f = 0$, and $\ell = 4$ and we use a degeneration argument to finish this case.

The statements and proofs of our main results would be simpler if more were known about the geometry of $\mathcal{M}_{g,f}$ and $\mathcal{H}_{g,f}$. For example, one could reduce to the case $f = 0$ if one knew that each irreducible component of $\mathcal{M}_{g,f}$ contained a component of $\mathcal{M}_{g,0}$. Even the number of irreducible components of $\mathcal{M}_{g,f}$ (or $\mathcal{H}_{g,f}$) is known only in special cases.

We also sketch a second proof of the main results that uses degeneration to the boundaries of $\mathcal{M}_{g,f}$ and $\mathcal{H}_{g,f}$, see Remark 3.9.

Remark 1.2. There is no information in Theorem 1.1 about the field of definition of the curves. In the literature, there are several results about curves with trivial automorphism group which are defined over finite fields. In [14] and [15], the author constructs an $\mathbb{F}_p$-curve C_0 of genus g with $\mathrm{Aut}_{\bar{\mathbb{F}}_p}(C_0) = \{1\}$ and a hyperelliptic $\mathbb{F}_p$-curve D_0 of genus g with $\mathrm{Aut}_{\bar{\mathbb{F}}_p}(D_0) \simeq \mathbb{Z}/2$. However, the p-ranks of C_0 and D_0 are not considered.

For $p = 2$ and $0 \leq f \leq g$, the author of [19] constructs a hyperelliptic $\mathbb{F}_2$-curve D_0 of genus g and p-rank f with $\mathrm{Aut}_{\bar{\mathbb{F}}_p}(D_0) \simeq \mathbb{Z}/2$. The analogous question for odd characteristic appears to be open. Furthermore, for all p it seems to be an open question whether there exists an $\mathbb{F}_p$-curve C_0 of genus g and p-rank f with $\mathrm{Aut}_{\bar{\mathbb{F}}_p}(C_0) = \{1\}$ [19, Question 1].

1.1. **Notation and background.** All objects are defined over an algebraically closed field k of characteristic $p > 0$. Let $\mathcal{M}_g$ be the moduli space of smooth projective connected curves of genus g, with tautological curve $\mathcal{C}_g \to \mathcal{M}_g$. Let $\mathcal{H}_g$ be the moduli space of smooth projective connected hyperelliptic curves of genus g, with tautological curve $\mathcal{D}_g \to \mathcal{H}_g$.

If C is a k-curve of genus g, the p-rank of C is the number $f \in \{0, \ldots, g\}$ such that $\mathrm{Jac}(C)[p](k) \cong (\mathbb{Z}/p)^f$. The p-rank is a discrete invariant which is lower semicontinuous in families. It induces a stratification of $\mathcal{M}_g$ by closed reduced subspaces $\mathcal{M}_{g,f}$ which parametrize curves of genus g with p-rank at most f. Similarly, let $\mathcal{H}_{g,f} \subset \mathcal{H}_g$ be the locus of hyperelliptic curves of genus g with p-rank at most f.

Recall that $\dim(\mathcal{M}_g) = 3g - 3$ and $\dim(\mathcal{H}_g) = 2g - 1$. Every irreducible component of $\mathcal{M}_{g,f}$ has dimension $2g - 3 + f$ by [7, Thm. 2.3]. Every irreducible component of $\mathcal{H}_{g,f}$ has dimension $g - 1 + f$ by [9, Thm. 1] when $p \geq 3$ and by [17, Cor. 1.3] when $p = 2$. In other words, the locus of curves of genus g and p-rank f has pure codimension $g - f$ in $\mathcal{M}_g$ and in $\mathcal{H}_g$.

Every irreducible component of $\mathcal{M}_{g,f}$ (and $\mathcal{H}_{g,f}$) has a geometric generic point η. Let $\mathcal{C}_\eta$ (resp. $\mathcal{D}_\eta$) denote the curve corresponding to the point η.

Let ℓ be prime. Let $\mathcal{M}_g^\ell \subset \mathcal{M}_g$ denote the locus of curves which admit an automorphism of order ℓ (after pullback by a finite cover of the base). The locus $\mathcal{M}_g^\ell$ is closed in $\mathcal{M}_g$. If D is a hyperelliptic curve, let ι denote the unique hyperelliptic involution of D. Then ι is in the center of $\mathrm{Aut}(D)$. Let $\mathcal{H}_g^\ell \subset \mathcal{H}_g$ denote the locus of hyperelliptic curves which admit a non-hyperelliptic automorphism of order ℓ. Let $\mathcal{H}_g^{4,\iota}$ denote the locus of hyperelliptic curves which admit an automorphism σ of order four such that $\sigma^2 = \iota$.

An Artin-Schreier curve is a curve that admits a structure as $\mathbb{Z}/p$-cover of the projective line. Let $\mathcal{AS}_g \subset \mathcal{M}_g$ denote the locus of Artin-Schreier curves of genus g and let $\mathcal{AS}_{g,f}$ denote its p-rank strata.

Unless stated otherwise, we assume $g \geq 3$ and $0 \leq f \leq g$.

2. The case of $\mathcal{M}_g$

2.1. **A dimension result.** Suppose Θ is an irreducible component of $\mathcal{M}_g^\ell$ with generic point ξ. Let Y be the quotient of $\mathcal{C}_\xi$ by a group of order ℓ. Let g_Y and f_Y be respectively the genus and p-rank of Y. Consider the $\mathbb{Z}/\ell$-cover $\phi : \mathcal{C}_\xi \to Y$. Let $B \subset Y$ be the branch locus of ϕ. If $\ell = p$, let j_b be the jump in the lower ramification filtration of ϕ at a branch point $b \in B$ [18, IV].

Lemma 2.1. *(i) If $\ell \neq p$, then $\dim(\Theta) \leq 2(g - g_Y)/(\ell - 1) + f_Y - 1$;*
(ii) If $\ell = p$, then $\dim(\Theta) \leq 2(g - g_Y)/(\ell - 1) + f_Y - 1 - \sum_{b \in B} \lfloor j_b/p \rfloor$.

Proof. Let $\phi : \mathcal{C}_\xi \to Y$ be as above, with branch locus $B \subset Y$. Because $g \geq 3$, if $g_Y = 1$ then $|B| > 0$. Let $\mathcal{M}_{g_Y, f_Y, |B|}$ be the moduli space of curves of genus g_Y and p-rank at most f_Y with $|B|$ marked points. Then $\dim(\mathcal{M}_{g_Y,f_Y,|B|}) = 2g_Y - 3 + f_Y + |B|$ if $g_Y \geq 1$. Also $\dim(\mathcal{M}_{0,0,|B|}) = |B| - 3$ if $|B| \geq 3$.

(i) Since $\phi : \mathcal{C}_\xi \to Y$ is tamely ramified, the curve $\mathcal{C}_\xi$ is determined by the quotient curve Y, the branch locus B, and ramification data which is discrete. Therefore, $\dim(\Theta) \leq \dim(\mathcal{M}_{g_Y,f_Y,|B|})$ if $g_Y \geq 1$ and $\dim(\Theta) \leq |B| - 3$ if $g_Y = 0$. By the Riemann-Hurwitz formula, $2g - 2 = \ell(2g_Y - 2) + |B|(\ell - 1)$. One can deduce that $|B| = 2(g - \ell g_Y)/(\ell - 1) + 2$ and the desired result follows.

(ii) By the Riemann-Hurwitz formula for wildly ramified covers [18, IV, Prop. 4],

$$2g - 2 = p(2g_Y - 2) + \sum_{b \in B} (j_b + 1)(p - 1).$$

For $b \in B$, let $\hat{\phi}_b : \hat{\mathcal{C}}_z \to \hat{Y}_b$ be the germ of the cover ϕ at the ramification point z above b. By [2, p.229], the dimension of the moduli space of covers $\hat{\phi}_b$ with ramification break j_b is $d_b = j_b - \lfloor j_b/p \rfloor$.

The local/global principle of formal patching (found, for example, in [2, Prop. 5.1.3]) implies $\dim(\Theta) \leq \dim(\mathcal{M}_{g_Y,f_Y,|B|}) + \sum_{b\in B} d_b$. Since $|B| + \sum_{b\in B} j_b = 2(g - pg_Y)/(p-1) + 2$, this simplifies to

$$\dim(\Theta) \leq 2(g-g_Y)/(p-1) + f_Y - 1 - \sum_{b\in B} \lfloor j_b/p \rfloor.$$

□

2.2. **No automorphism of order p.**

Lemma 2.2. *Suppose Γ is a component of $\mathcal{M}_{g,f}$ with geometric generic point η. Then $\mathcal{C}_\eta$ does not have an automorphism of order p.*

Proof. The strategy of the proof is to show that $\dim(\Gamma \cap \mathcal{M}_g^p) < \dim(\Gamma)$. Recall that $\dim(\Gamma) = 2g - 3 + f$ by [7, Thm. 2.3].

Let Θ be an irreducible component of $\Gamma \cap \mathcal{M}_g^p$, with geometric generic point ξ. Consider the resulting cover $\phi : \mathcal{C}_\xi \to Y$, which is either étale or wildly ramified. Let g_Y and f_Y be respectively the genus and p-rank of Y.

Suppose first that $g_Y = 0$. In other words, $\xi \in \mathcal{AS}_{g,f}$ and $\mathcal{C}_\xi$ is an Artin-Schreier curve. By [17, Lemma 2.6], $g = d(p-1)/2$ for some $d \in \mathbb{N}$. If $p = 2$, then $\dim(\mathcal{AS}_{g,f}) = g - 1 + f$ [17, Cor. 1.3]. If $p \geq 3$, then $\dim(\mathcal{AS}_{g,f}) \leq d - 1$ by [17, Thm. 1.1]. In either case, $\dim(\Theta) \leq \dim(\mathcal{AS}_{g,f}) < \dim(\Gamma)$ since $g \geq 3$.

Now suppose that $g_Y \geq 1$. If $p \geq 3$, Lemma 2.1(ii) implies that $\dim(\Theta) \leq g - g_Y + f_Y - 1 < 2g - 3 + f$.

If $p = 2$ and if $g_Y \geq 1$, let $|B|$ be the number of branch points of ϕ. By the Deuring-Shafarevich formula [5, Cor. 1.8], $f - 1 = 2(f_Y - 1) + |B|$. Lemma 2.1(ii) implies that $\dim(\Theta) \leq 2g - 2g_Y + (f - 1 - |B|)/2 - \sum_{b\in B} \lfloor j_b/2 \rfloor$. In particular, $\dim(\Theta) < 2g - 2g_Y + f/2$. So $\dim(\Theta) < 2g - 3 + f$ if $g_Y \geq 2$.

Suppose $p = 2$ and $g_Y = 1$. The hypothesis $g \geq 3$ implies that ϕ is ramified. So $|B| \geq 1$ and $j_b \geq 1$ for $b \in B$. Then $\dim(\Theta) < 2g - 3 + f/2$.

Thus $\dim(\Theta) < \dim(\Gamma)$ in all cases. This inequality implies that $\eta \notin \mathcal{M}_g^p$ and $\mathrm{Aut}(\mathcal{C}_\eta)$ does not contain an automorphism of order p. □

2.3. **The main result for $\mathcal{M}_{g,f}$.**

Theorem 2.3. *Suppose $g \geq 3$ and $0 \leq f \leq g$. Suppose η is the geometric generic point of an irreducible component Γ of $\mathcal{M}_{g,f}$. Then $\mathrm{Aut}(\mathcal{C}_\eta) = \{1\}$.*

Proof. By Lemma 2.2, $\mathrm{Aut}(\mathcal{C}_\eta)$ contains no automorphism of order p. Let $\ell \neq p$ be prime. Consider an irreducible component $\Theta \subset \Gamma \cap \mathcal{M}_g^\ell$. The result follows in any case where $\dim(\Theta) < \dim(\Gamma) = 2g - 3 + f$.

Let ξ be the geometric generic point of Θ. Let Y be the quotient of $\mathcal{C}_\xi$ by a group of order ℓ. Let g_Y and f_Y be the genus and p-rank of Y.

If $\ell \geq 3$, then Lemma 2.1(i) implies $\dim(\Theta) \leq g - g_Y + f_Y - 1$. Thus $\dim(\Theta) < 2g - 3 + f$ and $\mathcal{C}_\eta$ has no automorphism of order $\ell \geq 3$.

Suppose $\ell = 2$. If $g_Y = 0$, then $\mathcal{C}_\eta$ is hyperelliptic and in particular $\dim(\Theta) \leq \dim(\mathcal{H}_{g,f}) = g - 1 + f < 2g - 3 + f$. If $g_Y \geq 1$, then $\dim(\Theta) \leq 2g - 2g_Y + f_Y - 1$ which is less than $2g - 3 + f$ except when $g_Y = 1$ and $f = f_Y \leq 1$.

For the final case, when $\ell = 2$, $g_Y = 1$, and $f = f_Y$, Lemma 2.1 alone does not suffice to prove the claim. Let $\mathcal{M}_g^{2,Y}$ be the moduli space of curves of genus g which are $\mathbb{Z}/2$-covers of Y. It is the geometric fiber over the moduli point of Y of a map from a proper, irreducible Hurwitz space to $\mathcal{M}_1$ (see, e.g., [3, Cor. 6.12]). Therefore, $\mathcal{M}_g^{2,Y}$ is irreducible. Now $\xi \in \mathcal{M}_g^{2,Y} \cap \Gamma$. The strategy is to show that there exists $s \in \mathcal{M}_g^{2,Y}$ such that $f_s > f_Y$. From this, it follows that $\mathcal{M}_g^{2,Y} \cap \mathcal{M}_{g,f_Y}$ is a closed subset of $\mathcal{M}_g^{2,Y}$ of positive codimension. Then Θ is a closed subset of Γ of positive codimension, and the proof is complete.

To construct s, consider a $\mathbb{Z}/2$-cover $\psi_1 : Y \to \mathbb{P}^1$. If g is odd (resp. even), let $\psi_2 : X \to \mathbb{P}^1$ be a $\mathbb{Z}/2$-cover so that X has genus $(g-1)/2$ (resp. $g/2$) and so that the branch locus of ψ_2 contains exactly 2 (resp. 3) of the branch points of ψ_1. Since only 2 (resp. 3) of the branch points of ψ_2 are specified, one can suppose X is ordinary. Consider the fiber product $\psi : W \to \mathbb{P}^1$ of ψ_1 and ψ_2. Following the construction of [9, Prop. 3], W has genus g and p-rank at least $g/2$. Since W is a $\mathbb{Z}/2$-cover of Y, it corresponds to a point $s \in \mathcal{M}_g^{2,Y}$ with p-rank at least $f_Y + 1$. □

Here is the proof of part (i) of Theorem 1.1:

Corollary 2.4. *Suppose $g \geq 3$ and $0 \leq f \leq g$. There exists a smooth projective k-curve C of genus g and p-rank f with $\mathrm{Aut}(C) = \{1\}$.*

Proof. Let Γ be an irreducible component of $\mathcal{M}_{g,f}$, with geometric generic point η. Let $\Gamma' \subset \Gamma$ be the open, dense subset parametrizing curves with p-rank exactly f [7, Thm. 2.3]. By Theorem 2.3, $\mathrm{Aut}(\mathcal{C}_\eta) = 1$. The sheaf $\underline{\mathrm{Aut}}(\mathcal{C})$ is constructible on Γ', but there are only finitely many possibilities for the automorphism group of a curve of genus g. Therefore, there is a nonempty open subspace $U \subset \Gamma'$ such that, for each $s \in U(k)$, $\mathcal{C}_s$ has p-rank f and $\mathrm{Aut}(\mathcal{C}_s) = 1$. □

Corollary 2.5. *Let $g \geq 3$ and $0 \leq f \leq g$. There exists a principally polarized abelian variety (A, λ) over k of dimension g and p-rank f with $\mathrm{Aut}(A, \lambda) = \{\pm 1\}$.*

Proof. Let A be the Jacobian of the curve given in Corollary 2.4. The desired properties then follow from Torelli's theorem [13, Thm. 12.1]. □

3. The case of $\mathcal{H}_g$

Recall that $g \geq 3$ and $0 \leq f \leq g$.

3.1. **When $p = 2$.**

Lemma 3.1. *Let $p = 2$ and suppose η is the geometric generic point of a component Γ of $\mathcal{H}_{g,f}$. Then $\mathrm{Aut}(\mathcal{D}_\eta) \simeq \mathbb{Z}/2$.*

Proof. The automorphism group of a hyperelliptic curve always contains a (central) copy of $\mathbb{Z}/2$. Let $U \subset \Gamma$ be the subset parametrizing curves with automorphism group $\mathbb{Z}/2$. As in the proof of Corollary 2.4, U is open; it suffices to show that U is nonempty.

By [17, Cor. 1.3], $\mathcal{H}_{g,0}$ is irreducible of dimension $g-1$ when $p = 2$. For $g \geq 3$, there exists a hyperelliptic curve D_0 with p-rank 0 and $\mathrm{Aut}(D_0) \simeq \mathbb{Z}/2$ [19, Thm. 3]. The component Γ contains $\mathcal{H}_{g,0}$ by [17, Cor. 4.6]. Then U is non-empty since $U \cap \mathcal{H}_{g,0}$ is nonempty. □

3.2. **No automorphism of order p.** Suppose $p \geq 3$.

Lemma 3.2. *If $p|(2g+2)$ or $p|(2g+1)$, then $\dim \mathcal{H}_g^p = \lfloor (2g+2)/p \rfloor - 2$. Otherwise, $\mathcal{H}_g^p$ is empty.*

Proof. Suppose $s \in \mathcal{H}_g^p(k)$. There exists $\sigma \in \mathrm{Aut}(\mathcal{D}_s)$ of order p. Since ι and σ commute, σ descends to an automorphism of $\mathcal{D}_s/\langle\iota\rangle \simeq \mathbb{P}^1$. Let Z be the projective line $\mathcal{D}_s/\langle\sigma,\iota\rangle$. Then $\mathcal{D}_s \to Z$ is the fiber product of the hyperelliptic cover $\phi : \mathcal{D}_s/\langle\sigma\rangle \to Z$ and the $\mathbb{Z}/p$-cover $\psi : \mathcal{D}_s/\langle\iota\rangle \to Z$.

Since $\mathcal{D}_s/\langle\iota\rangle$ has genus zero, the cover ψ is ramified only at one point b and the jump j_b in the lower ramification filtration equals 1. After changing coordinates on $\mathcal{D}_s/\langle\iota\rangle$ and Z, the cover ψ is isomorphic to $c^p - c = x$.

If ϕ is not branched at ∞ then each branch point of ϕ lifts to p branch points of the cover $\mathcal{D}_s \to \mathcal{D}_s/\langle\iota\rangle$, and the branch locus of ϕ consists of $(2g+2)/p$ points. On the other hand, if ϕ is branched at ∞ then the branch locus of ϕ consists of $(2g+1)/p$ points. Therefore, if $\mathcal{H}_g^p(k)$ is nonempty, then either $p|(2g+1)$ or $p|(2g+2)$.

Moreover, any branch locus of size $\lfloor (2g+2)/p \rfloor$ uniquely determines such a cover ϕ. A point $s \in \mathcal{H}_g^p$ is determined by the branch locus of ϕ up to the action of affine linear transformations on Z. Thus $\dim(\mathcal{H}_g^p) = \lfloor (2g+2)/p \rfloor - 2$. □

Lemma 3.3. *Let η be the geometric generic point of a component of $\mathcal{H}_{g,f}$. Then $\mathrm{Aut}(\mathcal{D}_\eta)$ contains no automorphism of order p.*

Proof. By Lemma 3.2, $\mathcal{H}_g^p$ is either empty or of dimension $\lfloor (2g+2)/p \rfloor - 2$. If $g \geq 3$, then $\dim(\mathcal{H}_g^p) < g - 1 + f = \dim(\mathcal{H}_{g,f})$. Thus $\mathcal{D}_\eta$ does not have an automorphism of order p. □

3.3. **Extra automorphisms of order two and four.** Suppose $p \geq 3$. In this section, we show that the geometric generic point of any component of $\mathcal{H}_{g,f}$ parametrizes a curve with no extra automorphism of order two or four. The proof relies on degeneration and requires an analysis of curves of genus 2 and p-rank 0.

Lemma 3.4. *Suppose $p \geq 3$ and $g = 2$. If η is a geometric generic point of $\mathcal{H}_{2,0}$, then* $\mathrm{Aut}(\mathcal{D}_\eta) \simeq \mathbb{Z}/2$.

Proof. By [11, p.130], $\mathrm{Aut}(\mathcal{D}_\eta)/\langle\iota\rangle \simeq G$ where G is one of the following groups: $\{1\}$, $\mathbb{Z}/5$, $\mathbb{Z}/2$, S_3, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, D_{12}, S_4, or $\mathrm{PGL}_2(\mathbb{Z}/5)$. Let $T^G \subset \mathcal{H}_{2,0}$ be the sublocus parametrizing hyperelliptic curves D with $\mathrm{Aut}(D)/\langle\iota\rangle \simeq G$. Since every component of $\mathcal{H}_{2,0}$ has dimension one, it suffices to show that each T^G is zero-dimensional.

If $G = \mathbb{Z}/5$ and $s \in T^G(k)$, then the Jacobian of $\mathcal{D}_s$ has an action by $\mathbb{Z}/5$, and thus must be one of the two abelian surfaces with complex multiplication by $\mathbb{Z}[\zeta_5]$. Therefore, there exist at most two hyperelliptic curves D of genus 2 and p-rank 0 with $\mathrm{Aut}(D)/\langle\iota\rangle \simeq \mathbb{Z}/5$.

Now let G be any non-trivial group from the list other than $\mathbb{Z}/5$. A curve of genus two and p-rank zero is necessarily supersingular, and any supersingular hyperelliptic curve D of genus two with $\mathrm{Aut}(D)/\langle\iota\rangle \simeq G$ is superspecial by [11, Prop. 1.3]. Since there are only finitely many superspecial abelian surfaces, T^G is a proper closed subset of $\mathcal{H}_{2,0}$ for each $G \neq \{1\}$ on the list. Thus $\mathrm{Aut}(\mathcal{D}_\eta) \simeq \mathbb{Z}/2$. □

Lemma 3.5. *Suppose $p \geq 3$ and $g \geq 3$.*

(i) Then $\mathcal{H}_g^2$ is irreducible with dimension g;
(ii) there exists $s \in \mathcal{H}_g^2(k)$ such that $\mathcal{D}_s$ has p-rank at least 2;
(iii) and $\dim(\mathcal{H}_{g,0} \cap \mathcal{H}_g^2) < g - 1$.

Proof. Suppose $s \in \mathcal{H}_g^2(k)$. There is a Klein-four cover $\phi : \mathcal{D}_s \to \mathbb{P}^1_k$ such that ϕ is the fiber product of two hyperelliptic covers $\psi_i : C_i \to \mathbb{P}^1_k$ [9, Lemma 3].

If g is even, then one can assume that C_1 and C_2 both have genus $g/2$ and that the branch loci of ψ_1 and ψ_2 differ in a single point. If g is odd, then one can assume that C_1 has genus $(g+1)/2$, C_2 has genus $(g-1)/2$, and the branch locus of ψ_2 is contained in the branch locus of ψ_1 [9, Prop. 3]. In both cases, the third $\mathbb{Z}/2$-subquotient of $\mathcal{D}_s$ has genus zero. In particular, if f_s denotes the p-rank of $\mathcal{D}_s$ then $f_s = f_{C_1} + f_{C_2}$ [9, Cor. 2].

(i) This is found in [9, Cor. 1].
(ii) One can choose ψ_1 so that C_1 is ordinary. Then $f_s \geq \lceil \frac{g}{2} \rceil \geq 2$.
(iii) Suppose $s \in \mathcal{H}_{g,0}(k)$, so that $f_s = f_{C_1} = f_{C_2} = 0$. If g is even, then the parameter space for choices of ψ_1 has dimension $\dim(\mathcal{H}_{g/2,0}) = g/2 - 1$. For fixed ψ_1, the parameter space for choices of ψ_2 has dimension at most 1. Similarly, if g is odd, the parameter space for choices of ψ_1 has dimension $\dim(\mathcal{H}_{(g+1)/2,0}) = (g-1)/2$. For fixed ψ_1, there are at most finitely many possibilities for ψ_2. In either case $\dim(\mathcal{H}_{g,0} \cap \mathcal{H}_g^2) \leq \lfloor g/2 \rfloor < g - 1$.

□

Lemma 3.6. *Suppose $p \geq 3$ and $g \geq 3$. Then $\mathcal{H}_g^{4,\iota}$ is irreducible with dimension $g-1$ and its geometric generic point parametrizes a curve with positive p-rank.*

Proof. Suppose $s \in \mathcal{H}_g^{4,\iota}(k)$. Let σ be an automorphism of $\mathcal{D}_s$ of order 4 such that $\sigma^2 = \iota$. Consider the $\mathbb{Z}/4$-cover $\mathcal{D}_s \xrightarrow{\alpha} \mathbb{P}^1_x \xrightarrow{\beta} \mathbb{P}^1_z$. Then β is branched at two points and ramified at two points. Without loss of generality, one can suppose these are 0_x and ∞_x on $\mathbb{P}^1_x$ and 0_z and ∞_z on $\mathbb{P}^1_z$. This implies that the action of σ on $\mathbb{P}^1_x$ is given by $\sigma(x) = -x$.

The inertia groups of $\beta\circ\alpha$ above 0 and ∞ are subgroups of $\langle\sigma\rangle \simeq \mathbb{Z}/4$ which are not contained in $\langle\sigma^2\rangle$. Thus they each have order 4 and α is branched over 0_x and ∞_x. The other $2g$ branch points of α form orbits under the action of σ and one can denote them by $\{\pm\lambda_1, \ldots, \pm\lambda_g\}$. Without loss of generality, one can suppose $\lambda_g = 1$ and $\beta(\lambda_g) = 1$ and therefore $\mathcal{D}_s$ has an affine equation of the form $y^2 = x(x^2-1)\prod_{i=1}^{g-1}(x^2-\lambda_i^2)$.

Let $S = \mathbb{P}^1 - \{0, 1, \infty\}$. Let $\Delta \subset S^{g-1}$ be the strong diagonal consisting of all $(g-1)$-tuples $(x_1, \ldots, x_{g-1})$ so that $x_i = x_j$ for some $i \neq j$. Let $\Delta' \subset S^{g-1}$ consist of all $(g-1)$-tuples $(x_1, \ldots, x_{g-1})$ so that $x_i = -x_j$ for some $i \neq j$. There is a surjective morphism $\omega : (\mathbb{P}^1 - \{0, 1, \infty\})^{g-1} - (\Delta \cup \Delta') \to \mathcal{H}_g^{4,\iota}$, where ω sends $(\lambda_1, \ldots, \lambda_{g-1})$ to the isomorphism class of the curve with affine equation $y^2 = x(x^2-1)\prod_{i=1}^{g-1}(x^2-\lambda_i^2)$. Thus $\mathcal{H}_g^{4,\iota}$ is irreducible.

There are only finitely many fractional linear transformations fixing the set $\{\pm\lambda_1, \ldots, \pm\lambda_{g-1}, \pm 1, 0, \infty\}$. Thus ω is finite-to-one and $\dim(\mathcal{H}_g^{4,\iota}) = g-1$.

Suppose $g \geq 3$, and let η be the geometric generic point of $\mathcal{H}_g^{4,\iota}$. To finish the proof, it suffices to show that the p-rank of $\mathcal{D}_\eta$ is positive. Let $T = \mathrm{Spec}(k[[t]])$ and let $T' = \mathrm{Spec}(k((t)))$. Consider the image of the T'-point $(t\lambda_1, t\lambda_2, \lambda_3, \ldots, \lambda_{g-1})$ under ω. This gives a T'-point of $\mathcal{H}_g^{4,\iota} \subset \mathcal{H}_g$. The moduli space $\overline{\mathcal{H}}_g$ of stable hyperelliptic curves is proper, so the T'-point of $\mathcal{H}_g$ gives rise to a T-point of $\overline{\mathcal{H}}_g$. The special fiber of this T-point corresponds to a stable curve Y. The stable curve Y has two components Y_1 and Y_2 intersecting in an ordinary double point. Here Y_1 has genus 2 and affine equation $y_1^2 = x(x^2-\lambda_1^2)(x^2-\lambda_2^2)$, while Y_2 has genus $g-2$ and affine equation $y_2^2 = \prod_{i=3}^{g-1}(x^2-\lambda_i^2)$.

The moduli point $s \in \overline{\mathcal{H}}_g(k)$ of Y is in the closure of $\mathcal{H}_g^{4,\iota}$. The automorphism σ extends to Y, and stabilizes each of the two components Y_1 and Y_2. Therefore, the moduli point of Y_1 lies in $\mathcal{H}_2^{4,\iota}$. There is a one-parameter family of such curves Y_1 since one can vary the choice of λ_2. By Lemma 3.4, one can suppose that $f_{Y_1} \neq 0$. Now $f_Y = f_{Y_1} + f_{Y_2}$ by [4, Ex. 9.2.8]. Thus $f_Y \neq 0$. Since the p-rank can only decrease under specialization, and since s is in the closure of η, the p-rank of $\mathcal{D}_\eta$ is non-zero as well. □

3.4. **Main Result for $\mathcal{H}_{g,f}$.**

Theorem 3.7. *Suppose $g \geq 3$ and $0 \leq f \leq g$. If η is the geometric generic point of an irreducible component of $\mathcal{H}_{g,f}$, then* $\mathrm{Aut}(\mathcal{D}_\eta) \simeq \mathbb{Z}/2$.

Proof. Let Γ be the irreducible component of $\mathcal{H}_{g,f}$ whose geometric generic point is η. Suppose $\sigma \in \mathrm{Aut}(\mathcal{D}_\eta)$ has order ℓ with $\sigma \notin \langle \iota \rangle$. Then $p \geq 3$ by Lemma 3.1. Without loss of generality, one can suppose that either ℓ is prime or $\ell = 4$ with $\sigma^2 = \iota$.

If $\ell = 4$ and $\sigma^2 = \iota$, then $\mathcal{H}_g^{4,\iota}$ is irreducible with dimension $g-1$ by Lemma 3.6. This is strictly less than $\dim(\Gamma)$ unless $f = 0$. If $f = 0$, the two dimensions are equal but the geometric generic point of $\mathcal{H}_g^{4,\iota}$ corresponds to a curve of non-zero p-rank by Lemma 3.6. Thus $\mathcal{D}_\eta$ has no automorphism σ of order 4 with $\sigma^2 = \iota$.

If ℓ is prime, one can suppose that $\ell \neq p$ by Lemma 3.3. In [10, p.10], the authors use an argument similar to the proof of Lemma 3.2 to show that $\mathcal{H}_g^\ell$ is empty unless $\ell \mid (2g+2-i)$ for some $i \in \{0,1,2\}$; and if $\mathcal{H}_g^\ell$ is non-empty then its dimension is $d_{g,\ell} = -1 + (2g+2-i)/\ell$. If $d_{g,\ell} < \dim(\Gamma) = g+f-1$ then $\mathcal{D}_\eta$ cannot have an automorphism of order ℓ. This inequality is always satisfied when $\ell \geq 3$ since $g \geq 3$.

Suppose $\ell = 2$. Then $d_{g,\ell} < \dim(\Gamma)$ unless $f \leq 1$. If $f = 1$ then the two dimensions are equal. By Lemma 3.5, $\mathcal{H}_g^2$ is irreducible and contains the moduli point of a curve with p-rank at least two. Therefore, the component Γ of $\mathcal{H}_{g,1}$ is not the same as the unique irreducible component of $\mathcal{H}_g^2$.

Finally, suppose $\ell = 2$ and $f = 0$. By Lemma 3.5(iii), $\dim(\Gamma \cap \mathcal{H}_{g,0}) < g-1$. Thus $\eta \notin \mathcal{H}_g^2$, and $\mathrm{Aut}(\mathcal{D}_\eta) \simeq \mathbb{Z}/2$. □

Part (ii) of Theorem 1.1 now follows:

Corollary 3.8. *Suppose $g \geq 3$ and $0 \leq f \leq g$. There exists a smooth projective hyperelliptic k-curve D of genus g and p-rank f with* $\mathrm{Aut}(D) \simeq \mathbb{Z}/2$.

Proof. The result follows from Theorem 3.7, using the same argument that was used to deduce Corollary 2.4 from Theorem 2.3. □

Remark 3.9. The proof of the last statement of Lemma 3.6 uses the intersection of $\overline{\mathcal{H}}_g^{4,\iota}$ with the boundary component Δ_2 of $\overline{\mathcal{H}}_g$. More generally, one can give a different proof of the main results of this paper using induction. Here are the main steps of the inductive proof. If $g \geq 3$ and if $1 \leq i \leq g/2$, one can show that the closure of every component of $\mathcal{M}_{g,f}$ in $\overline{\mathcal{M}}_g$ intersects the boundary component Δ_i by [6, p.80], [12]. Points of Δ_i correspond to singular curves Y that have two components Y_1 and Y_2 of genera i and $g-i$ intersecting in an ordinary double point. Using a dimension argument, one can show that Y_1 and Y_2 are generically smooth and that their p-ranks f_1 and f_2 add up to f. If the generic point of a component of $\mathcal{M}_{g,f}$ parametrizes a curve with a nontrivial automorphism, another dimension argument shows that this automorphism stabilizes each of Y_1 and Y_2. This would imply that

the generic point of a component of $\mathcal{M}_{g-i,f_2}$ parametrizes a curve with non-trivial automorphism group, which would contradict the inductive hypothesis.

An analogous proof works for $\mathcal{H}_{g,f}$ when $p \geq 3$ using [7].

One can also use monodromy techniques to prove Corollary 2.5, see [1, App. 4.4].

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