

The integral monodromy of hyperelliptic and trielliptic curves

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Abstract

We compute the \mathbb{Z}/ℓ and \mathbb{Z}_ℓ monodromy of every irreducible component of the moduli spaces of hyperelliptic and trielliptic curves. In particular, we provide a proof that the \mathbb{Z}/ℓ monodromy of the moduli space of hyperelliptic curves of genus g is the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. We prove that the \mathbb{Z}/ℓ monodromy of the moduli space of trielliptic curves with signature (r, s) is the special unitary group $\mathrm{SU}_{(r,s)}(\mathbb{Z}/\ell \otimes \mathbb{Z}[\zeta_3])$.

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1 Introduction

If $C \rightarrow S$ is a relative smooth proper curve of genus $g \geq 1$ over an irreducible base, then the ℓ -torsion of the relative Jacobian of C encodes important information about the family. Suppose ℓ is invertible on S , and let $s \in S$ be a geometric point. The fundamental group $\pi_1(S, s)$ acts linearly on the fiber $\mathrm{Pic}^0(C)[\ell]_s \cong (\mathbb{Z}/\ell)^{2g}$, and one can consider the mod- ℓ monodromy representation associated to C :

$$\rho_{C \rightarrow S, \ell} : \pi_1(S, s) \rightarrow \mathrm{Aut}(\mathrm{Pic}^0(C)[\ell]_s) \cong \mathrm{GL}_{2g}(\mathbb{Z}/\ell).$$

Let $M_\ell(C \rightarrow S)$, or simply $M_\ell(S)$, be the image of this representation. If a primitive ℓ^{th} root of unity is defined globally on S , then $\mathrm{Pic}^0(C)[\ell]_s$ is equipped with a skew-symmetric form $\langle \cdot, \cdot \rangle$ and $M_\ell(C \rightarrow S) \subseteq \mathrm{Sp}(\mathrm{Pic}^0(C)[\ell]_s, \langle \cdot, \cdot \rangle) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. If $C \rightarrow S$ is a sufficiently general family of curves, then $M_\ell(C \rightarrow S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ [8].

In this paper, we compute $M_\ell(S)$ when S is an irreducible component of the moduli space of hyperelliptic or trielliptic curves and $C \rightarrow S$ is the tautological curve. The first result implies that there is no restriction on the monodromy group in the hyperelliptic case other than that it preserve the symplectic pairing. As a trielliptic curve is a $\mathbb{Z}/3$ -cover of a genus zero curve, the $\mathbb{Z}/3$ -action constrains the monodromy group to lie in a unitary group associated to $\mathbb{Z}[\zeta_3]$. The second result implies that this is the only additional restriction in the trielliptic case.

Theorem 3.4 *Let ℓ be an odd prime, and let k be an algebraically closed field in which 2ℓ is invertible. For $g \geq 1$, $M_\ell(\mathcal{H}_g \otimes k) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.*

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Theorem 3.8 *Let $\ell \geq 5$ be prime, and let k be an algebraically closed field in which 3ℓ is invertible. Let $\mathcal{T}^{\bar{\nu}}$ be any component of the moduli space of trielliptic curves of genus $g \geq 3$. Then $M_{\ell}(\mathcal{T}^{\bar{\nu}} \otimes k) \cong \text{SG}_{(r_{\nu}, s_{\nu})}(\mathbb{Z}/\ell)$ (where the latter is a unitary group defined in (3.4)).*

We also prove that the ℓ -adic monodromy group is $\text{Sp}_{2g}(\mathbb{Z}_{\ell})$ in the situation of Theorem 3.4 and is $\text{SG}_{(r_{\nu}, s_{\nu})}(\mathbb{Z}_{\ell})$ in the situation of Theorem 3.8.

Theorem 3.4 is an unpublished result of J.K. Yu and has already been used multiple times in the literature. In [7], Chavdarov assumes this result to show that the numerator of the zeta function of the typical hyperelliptic curve over a finite field is irreducible. Kowalski also uses this result in a similar fashion [21]. The first author used Theorem 3.4 to prove a conjecture of Friedman and Washington on class groups of quadratic function fields [2].

There are other results in the literature which are similar to Theorem 3.4 but which are not quite strong enough for the applications above. A'Campo [1, Th. 1] computes the topological monodromy of $\mathcal{H}_g \otimes \mathbb{C}$. On the arithmetic side, the \mathbb{Q}_{ℓ} , as opposed to \mathbb{Z}_{ℓ} , monodromy of \mathcal{H}_g is computed in [17, 10.1.16]. For purely group-theoretic reasons, one can deduce that $M_{\ell}(\mathcal{H}_g)$ is maximal for all sufficiently large ℓ [25]; see also [23]. However, the exceptional set of primes is difficult to calculate and depends on g .

There are results on \mathbb{Q}_{ℓ} -monodromy of cyclic covers of the projective line of arbitrary degree, e.g., [16, Sec. 7.9]. Also, in [12, 5.5], the authors prove that the projective representation $\mathbb{P}\rho_{C \rightarrow S, \ell}$ is surjective for many families of cyclic covers of the projective line. Due to a combinatorial hypothesis, their theorem does not apply to \mathcal{H}_g and applies to at most one component of the moduli space of trielliptic curves for each genus, see Remark 2.8. See also work of Zarhin, e.g., [31].

As an application, for all $p \geq 5$, we show using [6] that there exist hyperelliptic and trielliptic curves of every genus (and signature) defined over $\overline{\mathbb{F}}_p$ whose Jacobians are absolutely simple. In contrast with the applications above, these corollaries do not use the full strength of our results. Related work can be found in [15] where the authors produce curves with absolutely simple Jacobians over \mathbb{F}_p under the restriction $g \leq 3$.

Corollary 3.6 *Let $p \neq 2$ and let $g \in \mathbb{N}$. Then there exists a smooth hyperelliptic curve of genus g defined over $\overline{\mathbb{F}}_p$ whose Jacobian is absolutely simple.*

Corollary 3.11 *Let $p \neq 3$. Let $g \geq 3$ and let (r, s) be a trielliptic signature for g (Definition 2.10). Then there exists a smooth trielliptic curve defined over $\overline{\mathbb{F}}_p$ with genus g and signature (r, s) whose Jacobian is absolutely simple.*

Our proofs proceed by induction on the genus. The base cases for the hyperelliptic family rely on the fact that every curve of genus $g = 1, 2$ is hyperelliptic; the claim on monodromy follows from the analogous assertion about the monodromy of \mathcal{M}_g . The base case $g = 3$ for the trielliptic family involves a comparison with a Shimura variety of PEL type, namely, the Picard modular variety. An important step is to show that the monodromy group does not change in the base cases when one adds a labeling of the ramification points to the moduli problem.

The inductive step is similar to the method used in [9] and uses the fact that families of smooth hyperelliptic (trielliptic) curves degenerate to trees of hyperelliptic (triellip-

tic) curves of lower genus. The combinatorics of admissible degenerations require us to compute the monodromy exactly for the inductive step rather than up to isomorphism.

The inductive strategy using admissible degeneration developed here should work for other families of curves, especially for more general cyclic covers of the projective line. The difficulty is in the direct calculation of monodromy for the necessary base cases.

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2 Moduli spaces of curves with \mathbb{Z}/d -action

2.1 Stable \mathbb{Z}/d -covers of a genus zero curve

Let $G = \mathbb{Z}/d$ be a cyclic group of prime order d . Let $G^\times = G - \text{id}_G$. Let S be an irreducible scheme over $\text{Spec } \mathbb{Z}[1/d, \zeta_d]$. Let k be an algebraically closed field equipped with a map $\mathbb{Z}[1/d, \zeta_d] \rightarrow k$.

Let $\psi : C \rightarrow S$ be a semi-stable curve. In other words, ψ is flat and proper and the geometric fibers of C are connected, reduced curves whose only singularities are ordinary double points. If $s \in S$, let C_s denote the fiber of C over s . Let $\text{Sing}_S(C)$ be the set of $z \in C$ for which z is a singular point of the fiber $C_{\psi(z)}$.

A *mark* Ξ on C/S is a closed subscheme of $C - \text{Sing}_S(C)$ which is finite and étale over S . The *degree* of Ξ is the number of points in any geometric fiber of $\Xi \rightarrow S$. A marked semi-stable curve $(C/S, \Xi)$ is *stably marked* if every geometric fiber of C satisfies the following condition: for each irreducible component C_0 of genus zero, $\#(C_0 \cap (\text{Sing}_S(C) \cup \Xi)) \geq 3$.

Consider a G -action $\iota_0 : G \hookrightarrow \text{Aut}_S(C)$ on C . Denote the ramification locus of $C \rightarrow C/\iota_0(G)$ by R , and the smooth ramification locus by $R_{\text{sm}} = R - (R \cap \text{Sing}_S(C))$. We say that $(C/S, \iota_0)$ is a *stable G -curve* if C/S is a semi-stable curve; if $\iota_0 : G \hookrightarrow \text{Aut}_S(C)$ is an action of G ; if R_{sm} is a mark on C/S ; and if $(C/S, R_{\text{sm}})$ is stably marked.

If $z \in \text{Sing}_S(C)$, let $C_{z,1}$ and $C_{z,2}$ denote the two components of the formal completion of $C_{\psi(z)}$ at z . We say that a stable G -curve $(C/S, \iota_0)$ is *admissible* if the following conditions are satisfied for every geometric point $z \in R \cap \text{Sing}_S(C)$. First, $\iota_0(1)$ stabilizes each branch $C_{z,i}$; second, the characters of the action of ι_0 on the tangent spaces of $C_{z,1}$ and $C_{z,2}$ at z are inverses. The first condition implies that the *dihedral nodes* of [10, Def. 1.3] do not occur for $(C/S, \iota_0)$.

Throughout the paper, we suppose that $(C/S, \iota_0)$ is an admissible stable G -curve. We further assume throughout that $C/\iota_0(G)$ has arithmetic genus 0. Then $C/\iota_0(G)$ is also a stably marked curve [10, Prop. 1.4]. The mark on $C/\iota_0(G)$ is the smooth branch locus B_{sm} , which is the (reduced subscheme of) the image of R_{sm} under the morphism $C \rightarrow C/\iota_0(G)$. Let r be the degree of R_{sm} . By the Riemann-Hurwitz formula, the arithmetic genus of each fiber of C is $g = 1 - d + r(d - 1)/2$.

Remark 2.1. Suppose R is a complete discrete valuation ring with residue field k and fraction field K . Suppose $(C/K, \iota_0)$ is an admissible stable G -curve over K . As in [28, Section 6], see also [27], there is a finite extension R' of R such that, if $K' = \text{Frac } R'$, the pullback $(C_{K'}, \iota_0)$ has stable reduction. In other words, the G -curve $(C_{K'}, \iota_0)$ extends to an admissible stable G -curve $(C'/R', \iota_0)$. Let C'_k be the special fibre of C' . The extension

$R \rightarrow R'$ is necessary to guarantee that C'_k has only nodes as singularities and that the ramification divisor R_{sm} extends to a mark on C'_k . In particular, if z is a singular point of C'_k , then no point of R_{sm} specializes to z . As in [28, Lemma 6.3.5], this implies that the inertia group at z is cyclic and does not permute $C_{z,1}$ and $C_{z,2}$. Thus the cover (C'_k, ι_0) is admissible.

Let s be a geometric point of S with residue field k and let a be a point of the fiber $R_{\text{sm},s}$. Then G acts on the tangent space of C_s at a via a character $\chi_a : G \rightarrow k^\times$. In particular, there is a unique choice of $\gamma_a \in (\mathbb{Z}/d)^\times$ so that $\chi_a(1) = \zeta_d^{\gamma_a}$. We say that γ_a is the *canonical generator of inertia* at a . The *inertia type* of $(C/S, \iota_0)$ is the multiset $\{\gamma_a : a \in R_{\text{sm},s}\}$. It is independent of the choice of s . By Riemann's existence theorem, there exists a cover (C, ι_0) with inertia type $\{\gamma_a : a \in R_{\text{sm},s}\}$ if and only if $\sum_{a \in R_{\text{sm},s}} \gamma_a = 0 \in \mathbb{Z}/d$ [30, 2.13].

A labeling of a mark Ξ of degree r is a bijection η between $\{1, \dots, r\}$ and the irreducible components of Ξ . A labeling of an admissible stable G -curve $(C/S, \iota_0)$ is a labeling η of R_{sm} . There is an induced labeling $\eta_0 : \{1, \dots, r\} \rightarrow B_{\text{sm}}$.

If $(C/S, \iota_0, \eta)$ is a labeled G -curve, the *class vector* is the map of sets $\gamma : \{1, \dots, r\} \rightarrow G^\times$ such that $\gamma(i) = \gamma_{\eta(i)}$. We frequently write $\gamma = (\gamma(1), \dots, \gamma(r))$. If γ is a class vector, we denote its inertia type by $\bar{\gamma} : G^\times \rightarrow \mathbb{Z}_{\geq 0}$ where $\bar{\gamma}(h) = \#\gamma^{-1}(h)$ for all $h \in G$.

2.2 Moduli spaces

We define moduli functors on the category of schemes over $\text{Spec } \mathbb{Z}[1/d, \zeta_d]$ by describing their S -points:

$\overline{\mathcal{M}}_G$ parametrizes admissible stable G -curves $(C/S, \iota_0)$.

$\widetilde{\mathcal{M}}_G$ parametrizes labeled admissible stable G -curves $(C/S, \iota_0, \eta)$.

$\widetilde{\mathcal{M}}_{g,r}$ parametrizes triples $(C/S, \Xi, \eta)$ where C/S is a semi-stable curve of genus g , Ξ is a mark of degree r on C such that $(C/S, \Xi)$ is stably marked, and η is a labeling of Ξ .

Each functor is represented by an algebraic stack, and we use the same letter to denote both a moduli functor and its representing stack. For each of these moduli spaces \mathcal{M} , we let \mathcal{M}° denote the open substack whose objects parametrize smooth curves of the appropriate type. To work with fibers of the structural map $\mathcal{M} \rightarrow \text{Spec } \mathbb{Z}[1/d, \zeta_d]$ we write $\mathcal{M} \otimes k$ for $\mathcal{M} \times_{\text{Spec } \mathbb{Z}[1/d, \zeta_d]} \text{Spec } k$.

Lemma 2.2. *The moduli spaces $\widetilde{\mathcal{M}}_G$ and $\overline{\mathcal{M}}_G$ are smooth, proper Deligne-Mumford stacks over $\text{Spec } \mathbb{Z}[1/d, \zeta_d]$. The subspaces $\widetilde{\mathcal{M}}_G^\circ$ and $\overline{\mathcal{M}}_G^\circ$ are open and dense in $\widetilde{\mathcal{M}}_G$ and $\overline{\mathcal{M}}_G$, respectively.*

Proof. The moduli spaces $\widetilde{\mathcal{M}}_G$ and $\overline{\mathcal{M}}_G$ are algebraic stacks. Since the automorphism group scheme of a (labeled) admissible stable G -curve over an algebraically closed field is étale, $\overline{\mathcal{M}}_G$ and $\widetilde{\mathcal{M}}_G$ are Deligne-Mumford stacks [24, 8.1].

The local deformation problem for a semi-stable curve with G -action is formally smooth [10, 2.1], so $\overline{\mathcal{M}}_G$ is smooth. By [10, 2.3], the degree of the smooth ramification divisor of a stable admissible G -curve is locally constant, so $\widetilde{\mathcal{M}}_G$ is smooth, too.

We use the valuative criterion to show that these stacks are proper [24, 7.3]. Let \mathcal{O}_K be a complete discrete valuation ring with field of fractions K , and suppose $(C, \iota_0) \in \overline{\mathcal{M}}_G(K)$. By the stable reduction theorem, there exists a finite extension K'/K so that the curve C extends as a stable curve to $\mathcal{O}_{K'}$, as does the admissible G -action ι_0 . As in Remark 2.1, possibly after a further extension of the base, one may blow up the special fiber of $C/\mathcal{O}_{K'}$ to remove any dihedral nodes [10, p.195], so that no singular point of C is in the closure of R_{sm} and the resulting curve is an admissible stable G -curve. This shows that $\overline{\mathcal{M}}_G$ is proper. After a finite base change, a labeling of R_{sm} also extends uniquely. Thus, $\widetilde{\mathcal{M}}_G$ is also proper.

The openness and density of $\widetilde{\mathcal{M}}_G^\circ$ in $\widetilde{\mathcal{M}}_G$ and of $\overline{\mathcal{M}}_G^\circ$ in $\overline{\mathcal{M}}_G$ follow from the fact that an admissible stable G -curve is equivariantly smoothable [10, 2.2]. \square

Consequently, every connected component of $\overline{\mathcal{M}}_G$ or $\widetilde{\mathcal{M}}_G$ is irreducible.

Let $\gamma : \{1, \dots, r\} \rightarrow (\mathbb{Z}/d)^\times$ be a class vector of length $r = r(\gamma)$ for G . Let $g(\gamma) := 1 - d + r(\gamma)(d - 1)/2$. Let $\widetilde{\mathcal{M}}_G^\gamma$ be the substack of $\widetilde{\mathcal{M}}_G$ for which $(C/S, \iota_0, \eta)$ has class vector γ . Let $\overline{\mathcal{M}}_G^{\overline{\gamma}}$ be the substack of $\overline{\mathcal{M}}_G$ for which $(C/S, \iota_0)$ has inertia type $\overline{\gamma}$.

Lemma 2.3. *The moduli space $\widetilde{\mathcal{M}}_G^\gamma$ is irreducible.*

Proof. Since $\widetilde{\mathcal{M}}_G$ is proper and smooth over $\text{Spec } \mathbb{Z}[1/d, \zeta_d]$, it is sufficient to prove that $\widetilde{\mathcal{M}}_G^\gamma \otimes \mathbb{C}$ is irreducible [11, IV.5.10]. Since $\widetilde{\mathcal{M}}_G^{\gamma, \circ} \otimes \mathbb{C}$ is open and dense in $\widetilde{\mathcal{M}}_G^\gamma \otimes \mathbb{C}$, it suffices to prove that $\widetilde{\mathcal{M}}_G^{\gamma, \circ} \otimes \mathbb{C}$ is irreducible. Consider the functor $\beta_\gamma : \widetilde{\mathcal{M}}_G^\gamma \rightarrow \widetilde{\mathcal{M}}_{0,r(\gamma)}$. On S -points, this functor takes the isomorphism class of (C, ι_0, η) to the isomorphism class of $(C/\iota_0(G), B_{\text{sm}}, \eta_0)$, where η_0 is the induced labeling of B_{sm} . By Riemann's existence theorem [30, Section 2.2], $\beta_\gamma \otimes \mathbb{C} : \widetilde{\mathcal{M}}_G^{\gamma, \circ} \otimes \mathbb{C} \rightarrow \widetilde{\mathcal{M}}_{0,r}^\circ \otimes \mathbb{C}$ is an isomorphism. The statement follows since $\widetilde{\mathcal{M}}_{0,r}^\circ \otimes \mathbb{C}$ is irreducible. \square

If two class vectors γ and γ' yield the same inertia type, so that $\overline{\gamma} = \overline{\gamma}'$, then there is a permutation ω of $\{1, \dots, r\}$ such that $\gamma' = \gamma \circ \omega$. This relabeling of the branch locus yields an isomorphism

$$\widetilde{\mathcal{M}}_G^\gamma \xrightarrow{\sim} \widetilde{\mathcal{M}}_G^{\gamma \circ \omega}. \quad (2.1)$$

Suppose γ and γ' differ by an automorphism of G , so that there exists $\sigma \in \text{Aut}(G)$ such that $\gamma' = \sigma \circ \gamma$. This relabeling of the G -action yields an isomorphism

$$\widetilde{\mathcal{M}}_G^\gamma \xrightarrow{\sim} \widetilde{\mathcal{M}}_G^{\sigma \circ \gamma}. \quad (2.2)$$

Lemma 2.4. *The forgetful functor $\widetilde{\mathcal{M}}_G^\gamma \rightarrow \overline{\mathcal{M}}_G^{\overline{\gamma}}$ is étale and Galois.*

Proof. The map is étale since any admissible stable G -curve admits a labeling of R_{sm} étale-locally on the base. Moreover, the set of labelings of a fixed stable G -curve with rational smooth ramification locus is a torsor under the subgroup of $\text{Sym}(r)$ consisting of those ω for which $\gamma = \gamma \circ \omega$. Therefore, $\widetilde{\mathcal{M}}_G^\gamma \rightarrow \overline{\mathcal{M}}_G^{\overline{\gamma}}$ is Galois. \square

2.3 Degeneration

For $i = 1, 2, 3$, let γ_i denote a class vector with length r_i and let $g_i = g(\gamma_i)$. There is a clutching map

$$\widetilde{\mathcal{M}}_{g_1, r_1} \times \widetilde{\mathcal{M}}_{g_2, r_2} \rightarrow \widetilde{\mathcal{M}}_{g_1+g_2, r_1+r_2-2}$$

which is a closed immersion [19, 3.9]. On S -points, this map corresponds to gluing C_1 and C_2 together over S by identifying the last section of C_1 with the first section of C_2 .

There is a forgetful functor $\widetilde{\mathcal{M}}_G^\gamma \rightarrow \widetilde{\mathcal{M}}_{g(\gamma), r(\gamma)}$ taking the isomorphism class of $(C/S, \iota_0, \eta)$ to the isomorphism class of $(C/S, R_{\text{sm}}, \eta)$. This functor is finite-to-one since $\text{Aut}_S(C, \iota_0)$ is finite [8, 1.11]. The composition

$$\widetilde{\mathcal{M}}_G^{\gamma_1} \times \widetilde{\mathcal{M}}_G^{\gamma_2} \rightarrow \widetilde{\mathcal{M}}_{g_1, r_1} \times \widetilde{\mathcal{M}}_{g_2, r_2} \rightarrow \widetilde{\mathcal{M}}_{g_1+g_2, r_1+r_2-2}$$

allows us to glue two labeled G -curves $(C_i/S, \iota_{0,i}, \eta_i)$ together to obtain a labeled G -curve C/S with genus $g_1 + g_2$ and class vector

$$\gamma = (\gamma_1(1), \dots, \gamma_1(r_1 - 1), \gamma_2(2), \dots, \gamma_2(r_2)).$$

Moreover, C/S is equivariantly smoothable if and only if the G -action is admissible, i.e., if and only if $\gamma_1(r)$ and $\gamma_2(1)$ are inverses [10, 2.2]. In this situation, we say that (γ_1, γ_2) *deforms to* γ or that γ *degenerates to* (γ_1, γ_2) , and write

$$\widetilde{\mathcal{M}}_G^{\gamma_1} \times \widetilde{\mathcal{M}}_G^{\gamma_2} \longrightarrow \widetilde{\mathcal{M}}_G^\gamma.$$

The image $\Delta_{g(\gamma)}$ of this map is part of the boundary of $\widetilde{\mathcal{M}}_G^\gamma$.

The clutching maps admit a generalization to maps

$$\widetilde{\mathcal{M}}_{g_1, r_1} \times \widetilde{\mathcal{M}}_{g_2, r_2} \times \widetilde{\mathcal{M}}_{g_3, r_3} \rightarrow \widetilde{\mathcal{M}}_{g_1+g_2+g_3, r_1+r_2+r_3-4}.$$

Assume that (γ_1, γ_2) deforms to γ_L , that (γ_2, γ_3) deforms to γ_R , and that both (γ_L, γ_3) and (γ_1, γ_R) deform to γ . Then we have a commutative diagram of maps:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_G^{\gamma_1} \times \widetilde{\mathcal{M}}_G^{\gamma_2} \times \widetilde{\mathcal{M}}_G^{\gamma_3} & \rightarrow & \widetilde{\mathcal{M}}_G^{\gamma_1} \times \widetilde{\mathcal{M}}_G^{\gamma_R} \\ \downarrow & \searrow & \downarrow \\ \widetilde{\mathcal{M}}_G^{\gamma_L} \times \widetilde{\mathcal{M}}_G^{\gamma_3} & \longrightarrow & \widetilde{\mathcal{M}}_G^\gamma. \end{array} \tag{2.3}$$

In the situation above, suppose $g_1 = g_3 = 1$. The image $\Delta_{1,1}$ of $\widetilde{\mathcal{M}}_G^{\gamma_1} \times \widetilde{\mathcal{M}}_G^{\gamma_2} \times \widetilde{\mathcal{M}}_G^{\gamma_3}$ in $\widetilde{\mathcal{M}}_G^\gamma$ is part of the boundary of $\widetilde{\mathcal{M}}_G^\gamma$.

We say that a class vector γ *degenerates to* $\Delta_{1,1}$ if there are γ_1, γ_2 and γ_3 as in (2.3) so that $g_1 = g_3 = 1$. Furthermore, we say that an inertia type $\bar{\gamma}$ *degenerates to* $\Delta_{1,1}$ if there exists some class vector γ' with inertia type $\bar{\gamma}' = \bar{\gamma}$ so that γ' degenerates to $\Delta_{1,1}$. Using Equations (2.1) and (2.2), if $\bar{\gamma}$ degenerates to $\Delta_{1,1}$, then every component of $\widetilde{\mathcal{M}}_G^\gamma$ lying over $\overline{\mathcal{M}}_G^\gamma$ degenerates to $\Delta_{1,1}$ (although this may require identifying different ramification points of C_1, C_2 and C_3 in the clutching maps).

2.4 The case of hyperelliptic curves: $d = 2$

Let C/S be a $\mathbb{Z}/2$ -curve of genus g . Then C/S is hyperelliptic, and $\iota_0(1)$ is the hyperelliptic involution. Over every geometric point s of S , $C_s \rightarrow C_s/\iota_0(\mathbb{Z}/2)$ is ramified at $2g + 2$ smooth points.

There is a unique class vector $(1, \dots, 1)$ for $\mathbb{Z}/2$ of length $2g + 2$ for each $g \in \mathbb{N}$. By Lemma 2.3, there is a unique component of $\widetilde{\mathcal{M}}_{\mathbb{Z}/2}$ parametrizing labeled hyperelliptic curves of genus g . We denote this component by $\widetilde{\mathcal{H}}_g$. Similarly, $\overline{\mathcal{M}}_{\mathbb{Z}/2}$ has a unique component $\overline{\mathcal{H}}_g$ which parametrizes hyperelliptic curves of genus g . Let $\mathcal{H}_g = \overline{\mathcal{H}}_g^\circ$ denote the moduli space of smooth hyperelliptic curves of genus g .

Lemma 2.5. *If $g \geq 3$, then $\widetilde{\mathcal{H}}_g$ degenerates to $\Delta_{1,1}$.*

Proof. The class vector for $\widetilde{\mathcal{H}}_g$ is $\gamma = (1, \dots, 1)$ with length $2g + 2$. Let $\gamma_1 = \gamma_3 = (1, 1, 1, 1)$ and let $\gamma_2 = (1, \dots, 1)$ with length $2g - 2$. \square

Remark 2.6. When the boundary components Δ_1 and Δ_2 are distinct, the inductive argument in Theorem 3.4 can be revised to rely on monodromy groups only up to isomorphism. Unfortunately, when $g = 3$ one has $\Delta_1 = \Delta_2$, so that we must calculate monodromy groups exactly.

2.5 The case of trielliptic curves: $d = 3$

A *trielliptic* curve is a $\mathbb{Z}/3$ -curve $(C/S, \iota_0)$ so that $C/\iota_0(\mathbb{Z}/3)$ has genus zero. The curve C/S is sometimes called a cyclic trigonal curve. Over every geometric point s of S , the $\mathbb{Z}/3$ -cover $C_s \rightarrow C_s/\iota_0(\mathbb{Z}/3)$ is ramified at $g + 2$ smooth points.

2.5.1 Components of the trielliptic locus

The moduli space of trielliptic curves $(C/S, \iota_0)$ with C of genus g is not connected, even for fixed g . This reflects the different possibilities for the inertia type, or equivalently for the signature, which we now describe.

By Lemma 2.3, the components of $\widetilde{\mathcal{M}}_{\mathbb{Z}/3}$ parametrizing labeled trielliptic curves of genus g are in bijection with the maps $\gamma : \{1, \dots, g + 2\} \rightarrow (\mathbb{Z}/3)^\times$ such that $\sum \gamma(i) = 0 \in \mathbb{Z}/3$. We denote these components by $\widetilde{\mathcal{T}}^\gamma$, and their projections to $\overline{\mathcal{M}}_{\mathbb{Z}/3}$ by $\overline{\mathcal{T}}^\gamma$. Let $\mathcal{T}^\gamma = \overline{\mathcal{T}}^{\gamma^\circ}$ denote the moduli space of smooth trielliptic curves with inertia type $\bar{\gamma}$.

Lemma 2.7. *The set of possible inertia types $\bar{\gamma}$ for a trielliptic curve $(C/S, \iota_0)$ of genus g is in bijection with the set of pairs of integers (d_1, d_2) so that $d_1, d_2 \geq 0$, $d_1 + d_2 = g + 2$, and $d_1 + 2d_2 \equiv 0 \pmod{3}$.*

Proof. Let $d_1 = \bar{\gamma}(1)$ and $d_2 = \bar{\gamma}(2)$. The claim follows from earlier remarks. \square

Remark 2.8. The result [12, 5.5] applies to trielliptic curves only when $d_1 d_2 = 0$. These inertia types occur for curves of genus $3g_1 + 1$ having an equation of the form $y^3 = f(x)$ for some separable polynomial $f(x)$ of degree $3g_1 + 2$.

Let S be an irreducible scheme over $\text{Spec } \mathbb{Z}[1/3, \zeta_3]$. Then $\mathcal{O}_S \otimes_{\mathbb{Z}} \mathbb{Z}[1/3, \zeta_3] \cong \mathcal{O}_S \oplus \mathcal{O}_S$. We choose the isomorphism so that the first component has the given structure of \mathcal{O}_S as a $\mathbb{Z}[1/3, \zeta_3]$ -module.

Consider a trielliptic curve $(\psi : C \rightarrow S, \iota_0)$. The sheaf of relative one-forms $\psi_* \Omega_{C/S}^1$ is a locally free $\mathbb{Z}[1/3, \zeta_3] \otimes_{\mathbb{Z}} \mathcal{O}_S$ -module of some rank (r, s) , where $r + s = g$. We call (r, s) the *signature* of $(C/S, \iota_0)$. The signature is locally constant on S , so we may calculate it at any geometric point of S . If $(C/k, \iota_0)$ is a trielliptic curve, then $H^0(C, \Omega_C^1)$ decomposes as a direct sum $W_1 \oplus W_2$ where $\omega \in W_j$ if $\zeta_3 \circ \omega = \zeta_3^j \omega$. The signature of $(C/k, \iota_0)$ is $(r, s) = (\dim(W_1), \dim(W_2))$.

Lemma 2.9. *The signature and inertia type of a trielliptic curve of genus g are related as follows: $d_1 := \bar{\gamma}(1) = 2r - s + 1$ and $d_2 := \bar{\gamma}(2) = 2s - r + 1$. There exists a trielliptic curve $(C/k, \iota_0)$ of genus g with signature (r, s) if and only if $r, s \in \mathbb{Z}$, $r + s = g$, and $(g - 1)/3 \leq r, s \leq (2g + 1)/3$.*

Proof. Let $(C/k, \iota_0)$ be a trielliptic curve with inertia type $\bar{\gamma}$. Let $N = (d_1 + 2d_2)/3$. There is an equation for C of the form

$$y^3 = \prod_{i=1}^{d_1} (x - a_i) \prod_{j=1}^{d_2} (x - b_j)^2.$$

Consider the differential $\omega = \prod_{i=1}^{d_1} (x - a_i)^{n_i} \prod_{j=1}^{d_2} (x - b_j)^{n'_j} dx/y^m$ with $n_i, n'_j \geq 0$ and $m \geq 1$. By [20, Thm. 3], ω is holomorphic if and only if $3n_i \geq m - 2$, $3n'_j \geq 2m - 2$, and $mN \geq \sum_{i=1}^{d_1} n_i + \sum_{j=1}^{d_2} n'_j + 2$. Let $g(x) = \prod_{j=1}^{d_2} (x - b_j)$. Thus we have the following set of linearly independent holomorphic differentials:

$$\{dx/y, \dots, (x - a_1)^{N-2} dx/y, g(x) dx/y^2, \dots, (x - a_1)^{2N-2-d_2} g(x) dx/y^2\}.$$

This set is a basis of $H^0(C, \Omega_C^1)$ since its cardinality is $3N - d_2 - 2 = g$. Also, for any $h(x) \in k[x]$, $\zeta_3 \circ h(x) dx/y^m = \zeta_3^{-m} h(x) dx/y^m$. It follows that the signature of $(C/k, \iota_0)$ is $(r, s) = (g - N + 1, N - 1)$.

It follows that $d_1 = 2r - s + 1$ and $d_2 = 2s - r + 1$. Now $r, s \in \mathbb{Z}$ if and only if $d_1 + 2d_2 \equiv 0 \pmod{3}$. Also, $d_1 + d_2 = g + 2$ if and only if $r + s = g$. The conditions $d_1, d_2 \geq 0$ and $(g - 1)/3 \leq r, s \leq (2g + 1)/3$ are equivalent. The second claim then follows from Lemma 2.7. \square

Definition 2.10. Let $g \in \mathbb{N}$. A *trielliptic signature* for g is a pair (r, s) with $r, s \in \mathbb{Z}$, $r + s = g$, and $(g - 1)/3 \leq r, s \leq (2g + 1)/3$.

As in (2.2), if $(C/S, \iota_0)$ is a trielliptic curve then so is $(C/S, \iota'_0)$ where $\iota'_0(1) = \iota_0(2)$. Replacing ι_0 with ι'_0 exchanges the values of d_1 and d_2 and the values of r and s .

2.5.2 Degeneration of the trielliptic locus

We show that every component \tilde{T}^γ with $g(\gamma) \geq 4$ degenerates to $\Delta_{1,1}$.

Recall that there is a unique elliptic curve E which admits a $\mathbb{Z}/3$ -action ι_0 . The class vector of (E, ι_0) is either $(1, 1, 1)$ or $(2, 2, 2)$.

Proposition 2.11. *If $g(\gamma) \geq 4$, then \tilde{T}^γ degenerates to $\Delta_{1,1}$.*

Proof. Let $g = g(\gamma) \geq 4$ and let $\gamma : \{1, \dots, g+2\} \rightarrow (\mathbb{Z}/3)^\times$ be a class vector for $G = \mathbb{Z}/3$. It suffices to show that there exist class vectors γ_i for $i = 1, 2, 3$ with $g(\gamma_1) = g(\gamma_3) = 1$ as in (2.3). In particular, we require that γ_1 and γ_3 are either $(1, 1, 1)$ or $(2, 2, 2)$.

By Equations (2.1) and (2.2), we can reorder the values $\gamma(i)$ or replace each $\gamma(i)$ with $-\gamma(i)$. Thus it is sufficient to restrict attention to the inertia type $\bar{\gamma}$, or to the corresponding signature (r, s) by Lemma 2.9.

Suppose $\gamma_1 = \gamma_3 = (1, 1, 1)$. Then we need $\gamma_2 : \{1, \dots, g\} \rightarrow (\mathbb{Z}/3)^\times$ with $\gamma_2(1) = \gamma_2(g) = 2$ and $\bar{\gamma}_2(1) = \bar{\gamma}(1) - 4$ and $\bar{\gamma}_2(2) = \bar{\gamma}(2) + 2$. In other words, we need γ_2 to have four fewer points with inertia generator 1 and two more points with inertia generator 2 than γ does. This can be achieved by decreasing r by 2. Similarly, the case $\gamma_1 = \gamma_3 = (2, 2, 2)$ can be achieved by decreasing s by 2.

Suppose $\gamma_1 = (1, 1, 1)$ and $\gamma_3 = (2, 2, 2)$. Then we need $\gamma_2 : \{1, \dots, g\} \rightarrow (\mathbb{Z}/3)^\times$ with $\gamma_2(1) = 2$ and $\gamma_2(g) = 1$ and $\bar{\gamma}_2(1) = \bar{\gamma}(1) - 1$ and $\bar{\gamma}_2(2) = \bar{\gamma}(2) - 1$. In other words, we need γ_2 to have one fewer point with inertia generator 1 and one fewer point with inertia generator 2 than γ does. This can be done by decreasing both r and s by 1.

The next table shows that, for each inertia type $\bar{\gamma}$, there is a choice of $\gamma_1, \gamma_2, \gamma_3$ satisfying the numerical constraints.

Component γ with signature	degenerates so that γ_2 has signature	and γ_1 is	and γ_3 is	under this condition
(r, s)	$(r - 2, s)$	$(1, 1, 1)$	$(1, 1, 1)$	$r \geq 2$
(r, s)	$(r, s - 2)$	$(2, 2, 2)$	$(2, 2, 2)$	$s \geq 2$
(r, s)	$(r - 1, s - 1)$	$(1, 1, 1)$	$(2, 2, 2)$	$r \geq 1, s \geq 1$

□

Remark 2.12. When \tilde{T}^γ degenerates to both Δ_1 and Δ_2 , the inductive argument in Theorem 3.8 can be revised to rely on monodromy groups only up to isomorphism. Unfortunately, \tilde{T}^γ does not degenerate to Δ_2 when $\bar{\gamma}(1)\bar{\gamma}(2) = 0$. For $g(\gamma) \geq 4$, all other components \tilde{T}^γ degenerate to Δ_2 .

3 Monodromy groups

3.1 Definition of monodromy

Let $(X/S, \phi)$ be a principally polarized abelian scheme of relative dimension g over an irreducible base. If ℓ is a rational prime invertible on S , then the ℓ -torsion $X[\ell]$ of ℓ is an étale cover of S with geometric fiber isomorphic to $(\mathbb{Z}/\ell)^{2g}$. Let s be a geometric point of S . The fundamental group $\pi_1(S, s)$ acts linearly on the ℓ -torsion of X . This yields a representation

$$\rho_{X \rightarrow S, s, \ell} : \pi_1(S, s) \rightarrow \text{Aut}(X[\ell]_s) \cong \text{GL}_{2g}(\mathbb{Z}/\ell).$$

The cover $X[\ell] \rightarrow S$ both determines and is determined by the representation $\rho_{X \rightarrow S, s, \ell}$. The image of $\rho_{X \rightarrow S, s, \ell}$ is the *mod- ℓ monodromy* of $X \rightarrow S$ and we denote it by $M_\ell(X \rightarrow S, s)$,

or by $M_\ell(S, s)$ if the choice of abelian scheme is clear. The isomorphism class of the $M_\ell(S, s)$ is independent of the choice of base point s , and we denote it by $M_\ell(S)$.

Let X^\vee be the dual abelian scheme. There is a canonical pairing $X[\ell] \times X^\vee[\ell] \rightarrow \mu_{\ell, S}$, where $\mu_{\ell, S} := \mu_\ell \times S$ is the group scheme of ℓ^{th} roots of unity. The polarization ϕ induces an isomorphism $X \rightarrow X^\vee$, and thus a skew-symmetric pairing $X[\ell] \times X[\ell] \rightarrow \mu_{\ell, S}$. Because the polarization is defined globally, the image of monodromy $M_\ell(X \rightarrow S, s)$ is contained in the group of symplectic similitudes of $(X[\ell]_s, \langle \cdot, \cdot \rangle_\phi)$, which is isomorphic to $\text{GSp}_{2g}(\mathbb{Z}/\ell)$. Moreover, if a primitive ℓ^{th} root of unity exists globally on S , then $\pi_1(S, s)$ acts trivially on $\mu_{\ell, S}$ and $M_\ell(X \rightarrow S, s) \subseteq \text{Sp}(X[\ell]_s, \langle \cdot, \cdot \rangle_\phi) \cong \text{Sp}_{2g}(\mathbb{Z}/\ell)$.

Similarly, the cover $X[\ell^n] \rightarrow S$ defines a monodromy representation with values in $\text{Aut}(X[\ell^n]_s) \cong \text{GL}_{2g}(\mathbb{Z}/\ell^n)$. Taking the inverse limit over all n , we obtain a continuous representation on the Tate module of X ,

$$\rho_{X \rightarrow S, \mathbb{Z}_\ell, s} : \pi_1(S, s) \rightarrow \varprojlim_n \text{Aut}(X[\ell^n]_s) \cong \text{GL}_{2g}(\mathbb{Z}_\ell).$$

We denote the image of this representation by $M_{\mathbb{Z}_\ell}(X \rightarrow S, s)$, and its isomorphism class by $M_{\mathbb{Z}_\ell}(X \rightarrow S)$ or $M_{\mathbb{Z}_\ell}(S)$. Again, there is an inclusion $M_{\mathbb{Z}_\ell}(X \rightarrow S) \subseteq \text{GSp}_{2g}(\mathbb{Z}_\ell)$. If F is a field, let $F_{\ell^\infty} = F(\mu_{\ell^\infty}(\bar{F}))$. If S is an F -scheme, then $M_{\mathbb{Z}_\ell}(X \rightarrow S, s)/M_{\mathbb{Z}_\ell}(X \otimes \bar{F} \rightarrow S \otimes \bar{F}, s) \cong \text{Gal}(F_{\ell^\infty}/F)$. Finally, let $M_{\mathbb{Q}_\ell}(X \rightarrow S, s)$ be the Zariski closure of $M_{\mathbb{Z}_\ell}(X \rightarrow S, s)$ in $\text{GL}_{2g}(\mathbb{Q}_\ell)$.

Now suppose that $\psi : C \rightarrow S$ is a relative proper semi-stable curve. Let $\text{Pic}^0(C) := \text{Pic}_{C/S}^0$ be the neutral component of the relative Picard functor of C over S . Since C/S is semi-stable, $\text{Pic}^0(C)$ is a semiabelian scheme [4, 9.4.1]. Suppose that there is at least one geometric point s such that the fiber $\text{Pic}^0(C_s)$ is an abelian variety. (This is true if some C_s is a tree of smooth curves.) Then there is a nonempty open subscheme S^* of S such that $\text{Pic}^0(C|_{S^*})$ is an abelian scheme over S^* . We define the mod- ℓ and \mathbb{Z}_ℓ monodromy representations of C to be those of $\text{Pic}^0(C|_{S^*}) \rightarrow S^*$. (Alternatively, these may be constructed as the restrictions of $R^1\psi_*\mu_{\ell, S}$ and $R^1\psi_*\mu_{\ell^\infty, S}$ to the largest subscheme of S on which these sheaves are unramified.) Thus, $M_\ell(C \rightarrow S, s) = M_\ell(\text{Pic}^0(C|_{S^*}) \rightarrow S^*, s)$, and we denote this again by $M_\ell(S, s)$ if the curve is clear and by $M_\ell(S)$ if the base point is suppressed.

The moduli spaces $\overline{\mathcal{M}}_G$ and $\widetilde{\mathcal{M}}_G$ are Deligne-Mumford stacks, and we employ a similar formalism for étale covers of stacks [26]. Let \mathcal{S} be a connected Deligne-Mumford stack. The category of Galois étale covers of \mathcal{S} is a Galois category in the sense of Grothendieck, and thus there is an étale fundamental group of \mathcal{S} . More precisely, let $s \in \mathcal{S}$ be a geometric point. Then there is a group $\pi_1(\mathcal{S}, s)$ and an equivalence of categories between finite $\pi_1(\mathcal{S}, s)$ -sets and finite étale Galois covers of \mathcal{S} . If \mathcal{S} has a coarse moduli space S_{mod} , then $\pi_1(\mathcal{S}, s)$ is the extension of $\pi_1(S_{\text{mod}}, s)$ by a group which encodes the extra automorphism structure on the moduli space S_{mod} [26, 7.11]. If $X \rightarrow \mathcal{S}$ is a family of abelian varieties, we again let $M_\ell(X \rightarrow \mathcal{S}, s)$ be the image of $\pi_1(\mathcal{S}, s)$ in $\text{Aut}(X[\ell]_s)$.

Let \mathcal{C}^γ be the tautological labeled curve over $\widetilde{\mathcal{M}}_G^\gamma$. By the mod- ℓ or \mathbb{Z}_ℓ monodromy of $\widetilde{\mathcal{M}}_G^\gamma$ we mean that of $\mathcal{C}^\gamma \rightarrow \widetilde{\mathcal{M}}_G^\gamma$.

3.2 Degeneration and monodromy

In this section, we deduce information on monodromy groups from the degeneration of moduli spaces to the boundary. We will use this to prove inductively that the mod- ℓ monodromy groups of each $\tilde{\mathcal{H}}_g$ and each \tilde{T}^γ are as large as possible.

Lemma 3.1. *Let k be an algebraically closed field in which $d\ell$ is invertible. Suppose that the pair (γ_1, γ_2) deforms to γ , so that there is a clutching map*

$$\kappa : \tilde{\mathcal{M}}_G^{\gamma_1} \times \tilde{\mathcal{M}}_G^{\gamma_2} \longrightarrow \tilde{\mathcal{M}}_G^\gamma.$$

a. *There is a canonical isomorphism of sheaves on $(\tilde{\mathcal{M}}_G^{\gamma_1} \times \tilde{\mathcal{M}}_G^{\gamma_2}) \otimes k$,*

$$(\kappa \otimes k)^* \text{Pic}^0(\mathcal{C}^\gamma)[\ell] \cong \text{Pic}^0(\mathcal{C}^{\gamma_1})[\ell] \times \text{Pic}^0(\mathcal{C}^{\gamma_2})[\ell]. \quad (3.1)$$

b. *Suppose $s_i \in \tilde{\mathcal{M}}_G^{\gamma_i}(k)$ for $i = 1, 2$, and let $s = \kappa(s_1, s_2)$. After base change to k , there is a commutative diagram:*

$$\begin{array}{ccc} \text{M}_\ell(\tilde{\mathcal{M}}_G^{\gamma_1}, s_1) \times \text{M}_\ell(\tilde{\mathcal{M}}_G^{\gamma_2}, s_2) & \xrightarrow[\sim]{\alpha} & \text{M}_\ell(\kappa(\tilde{\mathcal{M}}_G^{\gamma_1} \times \tilde{\mathcal{M}}_G^{\gamma_2}), s) \hookrightarrow \text{M}_\ell(\tilde{\mathcal{M}}_G^\gamma, s) \\ \downarrow & & \searrow & \downarrow \\ \text{Aut}(\text{Pic}^0(\mathcal{C}^{\gamma_1})[\ell]_{s_1}) \times \text{Aut}(\text{Pic}^0(\mathcal{C}^{\gamma_2})[\ell]_{s_2}) & \xrightarrow{\delta} & \text{Aut}(\text{Pic}^0(\mathcal{C}^\gamma)[\ell]_s) \end{array}$$

Proof. Let $C \rightarrow S$ be a stable curve. Suppose that C is the union of two (not necessarily irreducible) proper, connected S -curves C_1 and C_2 which intersect along a unique section. If \mathcal{L} is a line bundle on C of degree zero, then for $i = 1, 2$ the restriction $\mathcal{L}|_{C_i}$ is a line bundle of degree zero on C_i [4, 9.1.2 and 9.2.13]. Thus, there is a morphism of group functors $(\kappa \otimes k)^* \text{Pic}^0(\mathcal{C}^\gamma) \rightarrow \text{Pic}^0(\mathcal{C}^{\gamma_1}) \times \text{Pic}^0(\mathcal{C}^{\gamma_2})$, which restricts to a morphism on ℓ -torsion. This is an isomorphism on stalks [4, 9.2.8], and thus an isomorphism of sheaves.

For part (b), the homomorphism δ is induced by the isomorphism in Equation (3.1). For $i = 1, 2$, let $C_i \rightarrow S_i$ denote the cover $\mathcal{C}^{\gamma_i} \otimes k \rightarrow \tilde{\mathcal{M}}_G^{\gamma_i} \otimes k$. To define α , consider the kernel of the representation $\rho_{C_i \rightarrow S_i, s_i, \ell}$. Since it is an open normal subgroup of the fundamental group, this kernel defines an irreducible Galois étale cover $Y_i \rightarrow S_i$. Let H_i be its Galois group. Then $Y_1 \times_k Y_2$ is an irreducible $H_1 \times H_2$ cover of $S_1 \times S_2$. Using this and part (a), the cover $\text{Pic}^0(\mathcal{C}^\gamma)[\ell]|_{\kappa(S_1 \times S_2)}$ is trivialized by a Galois extension with group $H_1 \times H_2$. Therefore, α is an isomorphism. \square

Lemma 3.2. *Let \mathbb{F} be a finite field of odd characteristic, and let V/\mathbb{F} be a g -dimensional vector space. Suppose that*

$$V = V_1 \oplus V_2 \oplus V_3, \quad (3.2)$$

and that $g_i := \dim V_i$ is positive for $i = 1, 2, 3$. Let H be a subgroup of $\text{GL}(V)$.

a. *If $H \subseteq \text{SL}(V)$ contains both $\text{SL}(V_1 \oplus V_2)$ and $\text{SL}(V_2 \oplus V_3)$, then $H = \text{SL}(V)$.*

b. Suppose that V is equipped with a nondegenerate pairing $\langle \cdot, \cdot \rangle$ and that for $i \neq j$, $\langle V_i, V_j \rangle = (0)$.

(i) If the pairing is symplectic, and if $H \subseteq \mathrm{Sp}(V)$ contains both $\mathrm{Sp}(V_1 \oplus V_2)$ and $\mathrm{Sp}(V_2 \oplus V_3)$, then $H = \mathrm{Sp}(V)$.

(ii) If the pairing is Hermitian, and if $H \subseteq \mathrm{SU}(V)$ contains both $\mathrm{SU}(V_1 \oplus V_2)$ and $\mathrm{SU}(V_2 \oplus V_3)$, then $H = \mathrm{SU}(V)$.

Hermitian pairings arise in the context of unitary groups; see Section 3.4.1.

Proof. We use (3.2) to identify $\mathrm{GL}(V_i)$ with a subgroup of $\mathrm{GL}(V)$.

Our proof of (a) is Lie-theoretic. Since $\mathrm{SL}(V)$ is split, the roots and Weyl group of $\mathrm{SL}(V)$ are the same as those of $\mathrm{SL}(V \otimes \overline{\mathbb{F}})$ [5, 1.18]. Moreover, $\mathrm{SL}(V)$ has a split BN-pair, and we show $H = \mathrm{SL}(V)$ by successively showing that H contains a maximal torus T of $\mathrm{SL}(V)$, the associated Weyl group, and all root subgroups.

Choose coordinates e_1, \dots, e_g so that V_1 is the span of $\{e_1, \dots, e_{g_1}\}$; V_2 is the span of $\{e_{g_1+1}, \dots, e_{g_1+g_2}\}$; and V_3 is the span of $\{e_{g_1+g_2+1}, \dots, e_g\}$.

First, H contains the maximal split torus T of $\mathrm{SL}(V)$ consisting of all diagonal matrices $\mathrm{diag}(\nu_1, \dots, \nu_g)$ such that $\prod_j \nu_j = 1$. To see this, choose j such that $e_j \in V_2$. Given $\nu = \mathrm{diag}(\nu_1, \dots, \nu_g) \in T$, we use the fact that $\nu_j = (\prod_{i \neq j} \nu_i)^{-1}$ to write

$$\nu = \mathrm{diag}(\nu_1, \dots, \nu_{j-1}, \left(\prod_{i=1}^{j-1} \nu_i\right)^{-1}, 1, \dots, 1) \cdot \mathrm{diag}(1, \dots, 1, \left(\prod_{i=j+1}^g \nu_i\right)^{-1}, \nu_{j+1}, \dots, \nu_g).$$

Second, H contains $N_{\mathrm{SL}(V)}(T)$, the normalizer of T in $\mathrm{SL}(V)$. It suffices to show $N_H(T)/T = N_{\mathrm{SL}(V)}(T)/T$. The normalizer $N_{\mathrm{SL}(V)}(T)$ is the set of matrices of determinant one with exactly one nonzero entry in each row and in each column, and the quotient $N_{\mathrm{SL}(V)}(T)/T$ is isomorphic to $\mathrm{Sym}(\{1, \dots, g\})$. Under this identification,

$$N_{\mathrm{SL}(V_1 \oplus V_2)}(T \cap \mathrm{SL}(V_1 \oplus V_2)) / (T \cap \mathrm{SL}(V_1 \oplus V_2)) = \mathrm{Sym}(\{1, \dots, g_1 + g_2\});$$

$$N_{\mathrm{SL}(V_2 \oplus V_3)}(T \cap \mathrm{SL}(V_2 \oplus V_3)) / (T \cap \mathrm{SL}(V_2 \oplus V_3)) = \mathrm{Sym}(\{g_1 + 1, \dots, g\}).$$

Since H contains $\mathrm{SL}(V_1 \oplus V_2)$ and $\mathrm{SL}(V_2 \oplus V_3)$, it contains $N_{\mathrm{SL}(V)}(T)$.

Third, let Δ be the canonical set of simple roots associated with $\mathrm{End}(V) \cong \mathrm{Mat}_n(V)$. For each $i \in \{1, \dots, g-1\}$ there is a root $\alpha_i \in \Delta$. The associated root group is the unipotent group U_{α_i} ; the nontrivial elements of U_{α_i} are unipotent matrices whose only nonzero offdiagonal entry is at $(i, i+1)$. Clearly, H contains each U_{α_i} .

Finally, the Weyl group $N_{\mathrm{SL}(V)}(T)/T$, acting on the set of simple roots Δ , generates the set of all roots Φ . Since H contains $N_{\mathrm{SL}(V)}(T)$, it therefore contains all root groups U_α where $\alpha \in \Phi$. By the Bruhat decomposition, any element of $\mathrm{SL}(V)$ is a product of a diagonal matrix and members of the various root groups. Therefore, $H = \mathrm{SL}(V)$.

For (b) part (i), after relabeling if necessary, we may assume that $V_1 \oplus V_2 \not\cong V_3$. Moreover, the hypothesis ensures that H contains $\mathrm{Sp}(V_3)$, and thus it contains $\mathrm{Sp}(V_1 \oplus V_2) \oplus \mathrm{Sp}(V_3)$. It is known that $\mathrm{Sp}(V_1 \oplus V_2) \oplus \mathrm{Sp}(V_3)$ is a maximal subgroup of $\mathrm{Sp}(V)$ [18, Thm. 3.2]. Since $\mathrm{Sp}(V_2 \oplus V_3) \not\subseteq \mathrm{Sp}(V_1 \oplus V_2) \oplus \mathrm{Sp}(V_3)$, the subgroup H must equal $\mathrm{Sp}(V)$.

The same argument proves (b) part (ii); as group-theoretic input, we use the fact [18, p.373] that $\mathrm{SU}(V_1 \oplus V_2) \oplus \mathrm{SU}(V_3)$ is a maximal subgroup of $\mathrm{SU}(V)$. \square

3.3 Monodromy of hyperelliptic curves

We show that the mod- ℓ monodromy of the tautological family of hyperelliptic curves of genus g is the full symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. Recall (Section 2.4) that for $g \in \mathbb{N}$, the moduli space $\tilde{\mathcal{H}}_g$ of labeled hyperelliptic curves of genus g is irreducible. We will use Lemma 3.3 to relate $M_\ell(\tilde{\mathcal{H}}_g)$ to $M_\ell(\overline{\mathcal{H}}_g)$.

Lemma 3.3. *Let ℓ be an odd prime. Let S be an irreducible scheme with a primitive $2\ell^{\mathrm{th}}$ root of unity, and let $(X \rightarrow S, \phi)$ be a principally polarized abelian scheme. Let $Y \rightarrow S$ be a Galois étale cover. Suppose that $M_\ell(X \rightarrow S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ and that the groups $\mathrm{Gal}(Y/S)$ and $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ have no common nontrivial quotients. Then $M_\ell(X \times_S Y \rightarrow Y) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.*

Proof. We equip $(\mathbb{Z}/\ell)^{2g}$ with the standard symplectic pairing $\langle \cdot, \cdot \rangle_{\mathrm{std}}$. Let

$$I_\ell := \mathrm{Isom}((X[\ell], \langle \cdot, \cdot \rangle_\phi), ((\mathbb{Z}/\ell)^{2g}, \langle \cdot, \cdot \rangle_{\mathrm{std}})).$$

The hypothesis on S implies that $I_\ell \rightarrow S$ is an étale Galois cover with group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. The hypothesis on $M_\ell(X \rightarrow S)$ implies that I_ℓ is irreducible.

To prove the lemma, we must show that $I_\ell \times_S Y$ is irreducible. Equivalently, we must show that $I_\ell \rightarrow S$ and $Y \rightarrow S$ are disjoint. Now, $I_\ell \times_S Y \rightarrow S$ is a (possibly reducible) étale Galois cover with group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell) \times \mathrm{Gal}(Y/S)$. If $Z \rightarrow S$ is any common quotient of $I_\ell \rightarrow S$ and $Y \rightarrow S$, then so is each conjugate $Z^\tau \rightarrow S$ for $\tau \in \mathrm{Gal}(I_\ell \times_S Y/S)$. Therefore, the compositum \tilde{Z} of all such conjugates Z^τ is also a common quotient of $I_\ell \rightarrow S$ and $Y \rightarrow S$. It thus suffices to show that there is no nontrivial Galois cover $\tilde{Z} \rightarrow S$ which is a quotient of $I_\ell \rightarrow S$ and $Y \rightarrow S$. This last claim is guaranteed by the group-theoretic hypothesis. \square

We compute the mod- ℓ monodromy of the moduli space of hyperelliptic curves.

Theorem 3.4. *Let ℓ be an odd prime, and let k be an algebraically closed field in which 2ℓ is invertible. For $g \geq 1$, $M_\ell(\tilde{\mathcal{H}}_g \otimes k) \cong M_\ell(\overline{\mathcal{H}}_g \otimes k) \simeq \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.*

Proof. Since $\overline{\mathcal{H}}_g \otimes k$ is open and dense in $\tilde{\mathcal{H}}_g \otimes k$, which is dominated by $\tilde{\mathcal{H}}_g \otimes k$, it suffices to show that $M_\ell(\tilde{\mathcal{H}}_g \otimes k) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.

Our proof is by induction on g . If $g = 1$ or $g = 2$, then every curve of genus g is hyperelliptic, so that $\overline{\mathcal{M}}_g \otimes k$ and $\overline{\mathcal{H}}_g \otimes k$ coincide. Therefore, $M_\ell(\tilde{\mathcal{H}}_g \otimes k) \cong M_\ell(\overline{\mathcal{M}}_g \otimes k)$. By [8, 5.15-5.16], $M_\ell(\overline{\mathcal{M}}_g \otimes k) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.

We conclude the base case $g = 1$ and $g = 2$ by applying Lemma 3.3 to the cover $\tilde{\mathcal{H}}_g \rightarrow \overline{\mathcal{H}}_g$. This cover is étale and Galois, with Galois group $\mathrm{Sym}(2g + 2)$. This group and $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ have no common nontrivial quotient. (To see this, recall that if ℓ is odd then the projective symplectic group $\mathrm{PSp}_{2g}(\mathbb{Z}/\ell)$ is simple, except that $\mathrm{PSp}_2(\mathbb{Z}/3) \cong A_4$. Neither $\mathrm{PSp}_{2g}(\mathbb{Z}/\ell)$ nor any quotient of A_4 is a nontrivial quotient of $\mathrm{Sym}(4)$ or $\mathrm{Sym}(6)$.) By Lemma 3.3, $M_\ell(\tilde{\mathcal{H}}_g \otimes k) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.

We now assume that $g \geq 3$ and that $M_\ell(\tilde{\mathcal{H}}_{g'} \otimes k) \cong \mathrm{Sp}_{2g'}(\mathbb{Z}/\ell)$ for $1 \leq g' < g$. By Lemma 2.5, $\tilde{\mathcal{H}}_g \otimes k$ degenerates to $\Delta_{1,1}$, so that there is a diagram (2.3):

$$\begin{array}{ccc} (\tilde{\mathcal{H}}_1 \times \tilde{\mathcal{H}}_{g-2} \times \tilde{\mathcal{H}}_1) \otimes k & \xrightarrow{\kappa_R} & (\tilde{\mathcal{H}}_1 \times \tilde{\mathcal{H}}_{g-1}) \otimes k \\ \downarrow \kappa_L & \searrow \kappa & \downarrow \\ (\tilde{\mathcal{H}}_{g-1} \times \tilde{\mathcal{H}}_1) \otimes k & \longrightarrow & \tilde{\mathcal{H}}_g \otimes k. \end{array}$$

Fix a base point $(s_1, s_2, s_3) \in (\tilde{\mathcal{H}}_1^\circ \times \tilde{\mathcal{H}}_{g-2}^\circ \times \tilde{\mathcal{H}}_1^\circ)(k)$, and let $s = \kappa(s_1, s_2, s_3)$. For $h \in \mathbb{N}$, let $C^h \rightarrow \tilde{\mathcal{H}}_h$ be the tautological labeled $\mathbb{Z}/2$ -curve of genus h . Let $V = \mathrm{Pic}^0(C^g)[\ell]_s$, and for $i = 1, 2, 3$ let $V_i = \mathrm{Pic}^0(C^{s_i})[\ell]_{s_i}$. Each of these is a \mathbb{Z}/ℓ -vector space equipped with a symplectic form. By Lemma 3.1(a), there is an isomorphism of symplectic \mathbb{Z}/ℓ -vector spaces $V \cong V_1 \oplus V_2 \oplus V_3$.

Let $\tilde{\mathcal{B}}_R$ (resp. $\tilde{\mathcal{B}}_L$) be the image of κ_R (resp. κ_L). Using the decomposition in Equation (3.1), we have inclusions:

$$\begin{aligned} M_\ell(\tilde{\mathcal{B}}_R \otimes k) &\subseteq \mathrm{Sp}(V_1) \oplus \mathrm{Sp}(V_2 \oplus V_3); \\ M_\ell(\tilde{\mathcal{B}}_L \otimes k) &\subseteq \mathrm{Sp}(V_1 \oplus V_2) \oplus \mathrm{Sp}(V_3). \end{aligned} \tag{3.3}$$

By the induction hypothesis, the inclusions in (3.3) are equalities. By Lemma 3.1(b), $M_\ell(\tilde{\mathcal{H}}_g \otimes k)$ contains $\mathrm{Sp}(V_1) \oplus \mathrm{Sp}(V_2 \oplus V_3)$ and $\mathrm{Sp}(V_1 \oplus V_2) \oplus \mathrm{Sp}(V_3)$. By Lemma 3.2b(i), $M_\ell(\tilde{\mathcal{H}}_g) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. \square

Corollary 3.5. *Let ℓ be an odd prime, and let k be an algebraically closed field in which 2ℓ is invertible. For $g \geq 1$, $M_{\mathbb{Z}_\ell}(\tilde{\mathcal{H}}_g \otimes k) \cong M_{\mathbb{Z}_\ell}(\mathcal{H}_g \otimes k) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$.*

Proof. By Theorem 3.4, $M_\ell(\tilde{\mathcal{H}}_g \otimes k) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. By construction, there is a surjection $M_{\mathbb{Z}_\ell}(\tilde{\mathcal{H}}_g \otimes k) \rightarrow M_\ell(\tilde{\mathcal{H}}_g \otimes k)$. Since the composition $M_{\mathbb{Z}_\ell}(\tilde{\mathcal{H}}_g \otimes k) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}_\ell) \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ is surjective, a standard group theory argument (e.g., [29, 1.3]) shows that $M_{\mathbb{Z}_\ell}(\tilde{\mathcal{H}}_g \otimes k) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$. As in the proof of Theorem 3.4, $M_{\mathbb{Z}_\ell}(\mathcal{H}_g \otimes k) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$ as well. \square

Corollary 3.6. *Let $p \neq 2$ and let $g \in \mathbb{N}$. Then there exists a smooth hyperelliptic curve of genus g defined over $\overline{\mathbb{F}}_p$ whose Jacobian is absolutely simple.*

Proof. Let ℓ be an odd prime distinct from p . By Corollary 3.5,

$$\mathrm{Sp}_{2g}(\mathbb{Z}_\ell) \cong M_{\mathbb{Z}_\ell}(\mathcal{H}_g \otimes \overline{\mathbb{F}}_p) \subseteq M_{\mathbb{Z}_\ell}(\mathcal{H}_g \otimes \mathbb{F}_p) \subseteq \mathrm{GSp}_{2g}(\mathbb{Z}_\ell).$$

Moreover, $M_{\mathbb{Z}_\ell}(\mathcal{H}_g \otimes \mathbb{F}_p) / M_{\mathbb{Z}_\ell}(\mathcal{H}_g \otimes \overline{\mathbb{F}}_p) \cong \mathrm{Gal}(\mathbb{F}_{p,\ell^\infty} / \mathbb{F}_p)$, which has finite index in $\mathrm{Aut}(\mu_{\ell^\infty}(\overline{\mathbb{F}}_p)) \cong \mathrm{GSp}_{2g}(\mathbb{Z}_\ell) / \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$. Therefore, $M_{\mathbb{Z}_\ell}(\mathcal{H}_g \otimes \mathbb{F}_p)$ is an open subgroup of $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$, and $M_{\mathbb{Q}_\ell}(\mathcal{H}_g \otimes \mathbb{F}_p) \cong \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$. The claim now follows from [6, Prop. 4]. \square

3.4 Monodromy of trielliptic curves

We now compute the monodromy groups of tautological families $\mathcal{C}^\gamma \rightarrow \tilde{\mathcal{T}}^\gamma$ of labeled trielliptic curves. The Jacobians of trielliptic curves admit an action by $\mathbb{Z}[\zeta_3]$. This places a constraint on $M_\ell(\tilde{\mathcal{T}}^\gamma)$; in Theorem 3.8, we show that this is the only constraint. We need more notation concerning $\mathbb{Z}[\zeta_3]$ -actions and unitary groups to describe the monodromy group precisely.

3.4.1 Unitary groups

Let S be an irreducible scheme over $\text{Spec } \mathbb{Z}[1/3, \zeta_3]$, and let $X \rightarrow S$ be an abelian scheme of relative dimension g equipped with an action $\iota : \mathbb{Z}[\zeta_3] \rightarrow \text{End}_S(X)$. Then $\text{Lie}(X)$ is a locally free $\mathbb{Z}[\zeta_3] \otimes \mathcal{O}_S$ -module of some rank (r, s) , where $r + s = g$. We call (r, s) the signature of the action of $\mathbb{Z}[\zeta_3]$ on X . If $(C/S, \iota_0)$ is a trielliptic curve, then $\text{Lie}(\text{Pic}^0(C))$ is the \mathcal{O}_S -linear dual of $\psi_* \Omega_{C/S}^1$ and the signature of $\text{Pic}^0(C)$ is the same as that of C .

Let $V_{(r,s)}$ be a free $\mathbb{Z}[\zeta_3]$ -module of rank g , equipped with a $\mathbb{Z}[\zeta_3]$ -linear pairing $\langle \cdot, \cdot \rangle$ of signature (r, s) . Let $\text{GU}_{(r,s)}$ be the $\mathbb{Z}[1/3, \zeta_3]$ -group scheme of similitudes of the pair $(V_{(r,s)}, \langle \cdot, \cdot \rangle)$, and let $\text{G}_{(r,s)}$ be the restriction of scalars of $\text{GU}_{(r,s)}$ to $\mathbb{Z}[1/3]$. Let $\text{SU}_{(r,s)} \subset \text{GU}_{(r,s)}$ be the sub-group scheme of elements of determinant one, and let $\text{SG}_{(r,s)}$ be the restriction of scalars of $\text{SU}_{(r,s)}$ to $\mathbb{Z}[1/3]$. Concretely, for any $\mathbb{Z}[1/3]$ -algebra R ,

$$\text{SG}_{(r,s)}(R) = \{ \tau \in \text{Aut}(V_{(r,s)} \otimes_{\mathbb{Z}[1/3]} R, \langle \cdot, \cdot \rangle) : \det(\tau) = 1 \}. \quad (3.4)$$

In the abstract, we used $\text{SU}_{(r,s)}(\mathbb{Z}/\ell \otimes \mathbb{Z}[\zeta_3])$ to denote $\text{SG}_{(r,s)}(\mathbb{Z}/\ell)$. The signature condition implies that $\text{SG}_{(r,s)}(\mathbb{R})$ is isomorphic to the complex special unitary group $\text{SU}(r, s)$.

The behavior of $\text{SG}_{(r,s)}$ at finite primes $\ell \geq 5$ depends on their splitting in $\mathbb{Z}[\zeta_3]$. Specifically, if ℓ is inert in $\mathbb{Z}[\zeta_3]$, then $\text{SG}_{(r,s)}(\mathbb{Z}/\ell) \cong \text{SU}_g(\mathbb{F}_{\ell^2})$. Alternatively, suppose that ℓ splits in $\mathbb{Z}[\zeta_3]$, and let λ and $\bar{\lambda}$ be the two primes of $\mathbb{Z}[\zeta_3]$ lying over ℓ . The factorization $\ell = \lambda \cdot \bar{\lambda}$ yields a factorization $\mathbb{Z}[\zeta_3] \otimes \mathbb{Z}/\ell \cong \mathbb{Z}/\ell \oplus \mathbb{Z}/\ell$. This induces a decomposition $V_{(r,s)} \otimes \mathbb{Z}/\ell \cong V_{(r,s)}(\lambda) \oplus V_{(r,s)}(\bar{\lambda})$, where $V_{(r,s)}(\lambda)$ and $V_{(r,s)}(\bar{\lambda})$ are g -dimensional \mathbb{Z}/ℓ -vector spaces. Moreover, the inner product $\langle \cdot, \cdot \rangle$ restricts to a perfect pairing between $V_{(r,s)}(\lambda)$ and $V_{(r,s)}(\bar{\lambda})$. Then $\text{SG}_{(r,s)}(\mathbb{Z}/\ell) \subset \text{Aut}(V_{(r,s)}(\lambda) \times V_{(r,s)}(\bar{\lambda}))$ is the image of $\text{SL}_g(\mathbb{Z}/\ell)$ embedded as $\tau \mapsto \tau \times ({}^t\tau)^{-1}$ (where ${}^t\tau$ denotes the transpose of τ). In either case, the isomorphism class of $\text{SG}_{(r,s)}(\mathbb{Z}/\ell)$ depends only on g and the splitting of ℓ in $\mathbb{Z}[\zeta_3]$, and not on the signature (r, s) .

3.4.2 Calculation of trielliptic monodromy

Let $\tilde{\mathcal{T}}^\gamma$ be any component of the moduli space of labeled trielliptic curves. The signature of the action of $\mathbb{Z}[\zeta_3]$ on the Lie algebra of the relative Jacobian is locally constant, and we denote it by (r_γ, s_γ) (see Lemma 2.9). The monodromy group $M_\ell(\tilde{\mathcal{T}}^\gamma \otimes k)$ must preserve both the symplectic pairing and the $\mathbb{Z}[\zeta_3]$ -action on the ℓ -torsion of $\text{Pic}^0(\mathcal{C}^\gamma \otimes k)$. This means that there is an inclusion $M_\ell(\tilde{\mathcal{T}}^\gamma \otimes k) \subseteq \text{SG}_{(r_\gamma, s_\gamma)}(\mathbb{Z}/\ell)$. We show that this is actually an isomorphism.

As a basis for induction, we compare the moduli space of trielliptic curves of genus three to the Picard modular variety, which is a component of a Shimura variety of PEL

type. Let $\mathcal{S}h_{(2,1)}$ be the moduli stack parametrizing data $(X/S, \iota, \phi)$ where $(X/S, \phi)$ is a principally polarized abelian scheme of relative dimension 3 and $\iota : \mathbb{Z}[\zeta_3] \rightarrow \text{End}(X)$ satisfies signature and involution constraints. Specifically, we require that $\text{Lie}(X)$ be a locally free $\mathbb{Z}[\zeta_3] \otimes \mathcal{O}_S$ -module of signature $(2, 1)$ and that ι take complex conjugation on $\mathbb{Z}[\zeta_3]$ to the Rosati involution on $\text{End}(X)$. Then $\mathcal{S}h_{(2,1)}$ is the Shimura variety associated to the reductive group $G_{(2,1)}$.

We also consider the moduli stack $\mathcal{S}h_{(2,1),\ell}$ parametrizing data $(X/S, \iota, \phi, \xi)$ where $(X/S, \iota, \phi)$ is as above and ξ is a principal level ℓ structure. More precisely, ξ is a $\mathbb{Z}[\zeta_3]$ -linear isomorphism $\xi : X[\ell] \rightarrow (V_{(2,1)} \otimes \mathbb{Z}/\ell) \otimes \mathcal{O}_S$ compatible with the given pairings. The forgetful functor $\mathcal{S}h_{(2,1),\ell} \rightarrow \mathcal{S}h_{(2,1)}$ induces a Galois cover of stacks, with covering group $\text{SG}_{(2,1)}(\mathbb{Z}/\ell)$ [13, 1.4]. Note that, as in [11, IV.6.1] but in contrast to [3, 22], we have implicitly chosen an isomorphism $\mu_\ell \rightarrow \mathbb{Z}/\ell$. If we had not chosen this isomorphism, the covering group would be $G_{(2,1)}(\mathbb{Z}/\ell)$; compare [11, IV.6.12].

Lemma 3.7. *Let $\ell \geq 5$ be prime, and let S be an irreducible scheme with a primitive $3\ell^{\text{th}}$ root of unity. Let $(X/S, \iota, \phi) \in \mathcal{S}h_{(2,1)}(S)$ be a principally polarized abelian scheme with $\mathbb{Z}[\zeta_3]$ -action. Let $Y \rightarrow S$ be an étale Galois cover. Suppose that $M_\ell(X \rightarrow S) \cong \text{SG}_{(2,1)}(\mathbb{Z}/\ell)$ and that the groups $\text{Gal}(Y/S)$ and $\text{SG}_{(2,1)}(\mathbb{Z}/\ell)$ have no common nontrivial quotients. Then $M_\ell(X \times_S Y \rightarrow Y) \cong \text{SG}_{(2,1)}(\mathbb{Z}/\ell)$.*

Proof. The proof is exactly the same as that of Lemma 3.3, except that it involves

$$I_{\mathbb{Z}[\zeta_3],\ell} := \text{Isom}_{\mathbb{Z}[\zeta_3]}((X[\ell], \langle \cdot, \cdot \rangle_\phi), (V_{2,1} \otimes \mathbb{Z}/\ell, \langle \cdot, \cdot \rangle)).$$

Then $I_{\mathbb{Z}[\zeta_3],\ell} \rightarrow S$ is étale and Galois with group $\text{SU}(V_{(2,1)} \otimes \mathbb{Z}/\ell) \simeq \text{SG}_{(2,1)}(\mathbb{Z}/\ell)$. The hypothesis on $M_\ell(X \rightarrow S)$ is equivalent to the irreducibility of $I_{\mathbb{Z}[\zeta_3],\ell}$. \square

Theorem 3.8. *Let $\ell \geq 5$ be prime, and let k be an algebraically closed field in which 3ℓ is invertible. Let $\tilde{\mathcal{T}}^\gamma$ be any component of the moduli space of labeled trielliptic curves. If $g(\gamma) \geq 3$, then $M_\ell(\tilde{\mathcal{T}}^\gamma \otimes k) \cong M_\ell(\mathcal{T}^{\bar{\gamma}} \otimes k) \cong \text{SG}_{(r_\gamma, s_\gamma)}(\mathbb{Z}/\ell)$.*

Proof. Since $\mathcal{T}^{\bar{\gamma}} \otimes k$ is open and dense in $\tilde{\mathcal{T}}^{\bar{\gamma}} \otimes k$, which is dominated by $\tilde{\mathcal{T}}^\gamma \otimes k$, it is sufficient to show $M_\ell(\tilde{\mathcal{T}}^\gamma \otimes k) \cong \text{SG}_{(r_\gamma, s_\gamma)}(\mathbb{Z}/\ell)$. We proceed by induction on $g(\gamma)$. The case $g(\gamma) = 3$ is supplied below in Lemma 3.9.

Now let $\tilde{\mathcal{T}}^\gamma$ be any component of $\tilde{\mathcal{T}}$ with $g = g(\gamma) \geq 4$, and suppose the result is true for all components $\tilde{\mathcal{T}}^{\gamma'}$ with $3 \leq g(\gamma') < g$. By Proposition 2.11, $\tilde{\mathcal{T}}^\gamma$ degenerates to $\Delta_{1,1}$. This means that there are class vectors $\gamma_1, \gamma_2, \gamma_3, \gamma_L$, and γ_R such that $g(\gamma_1) = g(\gamma_3) = 1$; $g(\gamma_2) = g - 2$; and there is a diagram (2.3):

$$\begin{array}{ccc} (\tilde{\mathcal{T}}^{\gamma_1} \times \tilde{\mathcal{T}}^{\gamma_2} \times \tilde{\mathcal{T}}^{\gamma_3}) \otimes k & \xrightarrow{\kappa_R} & (\tilde{\mathcal{T}}^{\gamma_1} \times \tilde{\mathcal{T}}^{\gamma_R}) \otimes k \\ \downarrow \kappa_L & \searrow \kappa & \downarrow \\ (\tilde{\mathcal{T}}^{\gamma_L} \times \tilde{\mathcal{T}}^{\gamma_3}) \otimes k & \longrightarrow & \tilde{\mathcal{T}}^\gamma \otimes k. \end{array}$$

Let $(s_1, s_2, s_3) \in (\tilde{\mathcal{T}}^{\gamma_1, \circ} \times \tilde{\mathcal{T}}^{\gamma_2, \circ} \times \tilde{\mathcal{T}}^{\gamma_3, \circ})(k)$, and let $s = \kappa(s_1, s_2, s_3)$. Let $V = \text{Pic}^0(\mathcal{C}^\gamma)[\ell]_s$, and for $i = 1, 2, 3$ let $V_i = \text{Pic}^0(\mathcal{C}^{\gamma_i})[\ell]_{s_i}$. Then $V \cong V_1 \oplus V_2 \oplus V_3$ is a decomposition of

V as a Hermitian $(\mathbb{Z}[\zeta_3] \otimes \mathbb{Z}/\ell)$ -module (Lemma 3.1(a)). Let $\tilde{\mathcal{B}}_R$ be the image of κ_R , and let $\tilde{\mathcal{B}}_L$ be the image of κ_L . Lemma 3.1(b) shows that $M_\ell(\tilde{\mathcal{T}}^\gamma \otimes k, s)$ contains $M_\ell(\tilde{\mathcal{B}}_L \otimes k)$ and $M_\ell(\tilde{\mathcal{B}}_R \otimes k)$; the inductive hypothesis shows that these are $SU(V_1 \oplus V_2)$ and $SU(V_2 \oplus V_3)$, respectively.

If ℓ is inert in $\mathbb{Z}[\zeta_3]$, Lemma 3.2b(ii) implies that $M_\ell(\tilde{\mathcal{T}}^\gamma \otimes k) \cong SU_g(\mathbb{F}_{\ell^2}) \cong SG_{(r_\gamma, s_\gamma)}(\mathbb{Z}/\ell)$.

Otherwise, if $\ell = \lambda \cdot \bar{\lambda}$ is split in $\mathbb{Z}[\zeta_3]$, let $V(\lambda)$ be the eigenspace of V corresponding to λ , and define $V_i(\lambda)$ analogously for $i = 1, 2, 3$. In this case, $SG_{(r_\gamma, s_\gamma)}(\mathbb{Z}/\ell)$, $SU(V)$ and $SL(V(\lambda))$ are isomorphic. By the inductive hypothesis, the projection of $M_\ell(\tilde{\mathcal{T}}^\gamma \otimes k)$ to $SL(V(\lambda))$ contains $SL(V_1(\lambda) \oplus V_2(\lambda))$ and $SL(V_2(\lambda) \oplus V_3(\lambda))$. By Lemma 3.2(a), we see that $M_\ell(\tilde{\mathcal{T}}^\gamma \otimes k) \cong SL(V(\lambda)) \cong SG_{(r_\gamma, s_\gamma)}(\mathbb{Z}/\ell)$. \square \square

Lemma 3.9. *Theorem 3.8 is true for $g(\gamma) = 3$.*

Proof. We start by computing the monodromy group of the Picard modular variety $Sh_{(2,1)}$. Both $Sh_{(2,1)} \otimes \mathbb{C}$ and $Sh_{(2,1), \ell} \otimes \mathbb{C}$ are arithmetic quotients of the complex ball, thus irreducible [13, 1.4]. Moreover, the existence of smooth arithmetic compactifications [22, Section 3] [3, 1.3.13] and Zariski's connectedness theorem (see [11, IV.5.10]) imply that $Sh_{(2,1)} \otimes k$ and $Sh_{(2,1), \ell} \otimes k$ are also irreducible. The irreducibility of the cover $Sh_{(2,1), \ell} \otimes k \rightarrow Sh_{(2,1)} \otimes k$ implies that the fundamental group of $Sh_{(2,1)} \otimes k$ acts transitively on the set of Hermitian $\mathbb{Z}[\zeta_3] \otimes \mathbb{Z}/\ell$ -bases for the ℓ -torsion of the tautological abelian scheme over $Sh_{(2,1)} \otimes k$. Therefore, $M_\ell(Sh_{(2,1)} \otimes k) \cong SG_{(2,1)}(\mathbb{Z}/\ell)$.

We now consider $\tilde{\mathcal{T}}^\gamma \otimes k$. Possibly after relabeling (2.2), by Lemma 2.9 we may assume that $(r_\gamma, s_\gamma) = (2, 1)$ and that the inertia type $\bar{\gamma}$ is $\{1, 1, 1, 1, 2\}$. Moreover, since $\dim Sh_{(2,1)} \otimes k = 2$, the Torelli map gives an identification of $\mathcal{T}^{\bar{\gamma}}$ with an open subset of $Sh_{(2,1)}$. Thus $M_\ell(\bar{\mathcal{T}}^{\bar{\gamma}} \otimes k) \cong M_\ell(Sh_{(2,1)} \otimes k) \cong SG_{(2,1)}(\mathbb{Z}/\ell)$.

By Lemma 2.4, $\tilde{\mathcal{T}}^\gamma \rightarrow \bar{\mathcal{T}}^{\bar{\gamma}}$ is Galois with group $\text{Sym}(4)$. Since $\ell \geq 5$, the quotient of $SG_{(2,1)}(\mathbb{Z}/\ell)$ by its center is simple. Thus, $\text{Sym}(4)$ and $SG_{(2,1)}(\mathbb{Z}/\ell)$ have no common nontrivial quotient, and $M_\ell(\tilde{\mathcal{T}}^\gamma \otimes k) \cong SG_{(2,1)}(\mathbb{Z}/\ell)$ by Lemma 3.7. \square \square

Corollary 3.10. *Let ℓ be an odd prime, and let k be an algebraically closed field in which 3ℓ is invertible. Let $\tilde{\mathcal{T}}^\gamma$ be any component of the moduli space of trielliptic curves. If $g(\gamma) \geq 3$, then $M_{\mathbb{Z}_\ell}(\tilde{\mathcal{T}}^\gamma \otimes k) \cong M_{\mathbb{Z}_\ell}(\bar{\mathcal{T}}^{\bar{\gamma}} \otimes k) \cong SG_{(r_\gamma, s_\gamma)}(\mathbb{Z}_\ell)$.*

Proof. The proof is parallel to that of Corollary 3.5; any subgroup of $SG_{(r_\gamma, s_\gamma)}(\mathbb{Z}_\ell)$ which surjects onto $SG_{(r_\gamma, s_\gamma)}(\mathbb{Z}/\ell)$ is all of $SG_{(r_\gamma, s_\gamma)}(\mathbb{Z}_\ell)$. \square

Corollary 3.11. *Let $p \neq 3$. Let $g \geq 3$ and let (r, s) be a trielliptic signature for g (Definition 2.10). Then there exists a smooth trielliptic curve defined over $\bar{\mathbb{F}}_p$ with genus g and signature (r, s) whose Jacobian is absolutely simple.*

Proof. Let $\mathbb{F} = \mathbb{F}_{p^2}$. Let ℓ be an odd prime distinct from p which is inert in $\mathbb{Z}[\zeta_3]$, and let $K_\ell = \mathbb{Q}_\ell(\zeta_3)$. By Lemma 2.9, there is a class vector γ whose inertia type has signature (r, s) . As in the proof of Corollary 3.6, using Corollary 3.10 one sees that $M_{\mathbb{Q}_\ell}(\tilde{\mathcal{T}}^\gamma \otimes \mathbb{F}) \cong GU_{(r,s)}(K_\ell)$.

Let F be a CM field of degree $[F : \mathbb{Q}] = 2g$ which contains $\mathbb{Q}(\zeta_3)$ and is inert at ℓ , and let $F_\ell = F \otimes \mathbb{Q}_\ell$. There is a torus $H \subset GU_{(r,s)}(K_\ell)$ isomorphic to F_ℓ^\times . This torus is

maximal since $[F : \mathbb{Q}] = 2g$. The quotient of H by the center of $\mathrm{GU}_{(r,s)}(K_\ell)$ is isogenous to the kernel J of the norm map $F_\ell^\times \rightarrow \mathbb{Q}_\ell^\times$. Since F_ℓ is a field, J is anisotropic and H is elliptic. Finally, H acts irreducibly on the Tate module of the Jacobian of the tautological curve C^γ , since $T_\ell(\mathrm{Pic}^0(C^\gamma)) \otimes \mathbb{Q}_\ell$ is a one-dimensional F_ℓ -vector space. The result then follows from [6, Remark 5(i)]. \square

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