On the Moduli Space of Klein Four Covers of the Projective Line

D. Glass, R. Pries [a]

Darren Glass
Department of Mathematics
Columbia University
New York, NY 10027
glass@math.columbia.edu

Rachel Pries
Department of Mathematics
Colorado State University
Fort Collins, CO 80523
pries@math.colostate.edu

We study the sublocus $\mathcal{H}_{g,n}$ of $\mathcal{M}_g$ whose points correspond to curves which are $(\mathbb{Z}/2\mathbb{Z})^n$ covers of the projective line. In the case that $n = 2$, we describe a set of irreducible components of $\mathcal{H}_{g,2}$ and all of the intersections between these components. Unlike the case of the hyperelliptic locus $\mathcal{H}_{g,1}$ which is well-known to be connected, we show that $\mathcal{H}_{g,n}$ is not connected when $n = 2$ and $g \geq 4$ or when $n = 3$ and $g \geq 5$.

1. Introduction

In this paper, we study the locus of smooth projective curves of genus g which have multiple commuting involutions. More precisely, we study the sublocus $\mathcal{H}_{g,n}$ of $\mathcal{M}_g$ whose points correspond to curves which are $(\mathbb{Z}/2\mathbb{Z})^n$ covers of the projective line. This locus is a generalization of the hyperelliptic locus $\mathcal{H}_g$. In the case that $n = 2$, we describe a set of irreducible components of $\mathcal{H}_{g,2}$ and Theorem 9 describes all of the intersections between these components. Unlike the case of $\mathcal{H}_g$ ($n = 1$) which is well-known to be connected, we show that $\mathcal{H}_{g,n}$ is not connected when $n = 2$ and $g \geq 4$ or

[a]The second author was partially supported by NSF grant DMS-04-00461.

when $n = 3$ and $g \geq 5$, Propositions 6 and 14. In particular, $\mathcal{H}_{g,n} \cap \mathcal{H}_g$ is a proper non-empty connected component of $\mathcal{H}_{g,n}$ in these cases.

This paper can be viewed as part of a larger project to understand the sublocus S_g of the moduli space $\mathcal{M}_g$ of smooth curves of genus g whose points correspond to curves with automorphisms. For $g \geq 4$, S_g is the singular locus of $\mathcal{M}_g$, [8]. In [2], Cornalba describes the irreducible components of S_g. The generic point of one of these components corresponds to a curve whose automorphism group is cyclic of prime order.

It is not known in general how the irreducible components of S_g intersect. One approach is to describe all Hurwitz loci and the inclusions between them using algorithms that measure the action of the braid group, see for example [6], [7]. The approach in this paper is complementary in that we restrict to one choice of automorphism group and search for results which are true for all genera.

We would like to thank T. Shaska for inviting us to contribute to this volume.

2. Hurwitz Spaces

Let k be an algebraically closed field of characteristic p where $p = 0$ or $p > 2$. Let G be an elementary abelian 2-group of order 2^n. In Section 2.1, we recall results on the moduli space $H_{g,n}$ parametrizing G-Galois covers $f : X \to \mathbb{P}^1_k$ where X is a smooth projective k-curve of genus g. In Section 2.2, we describe the connected components of $H_{g,n}$.

2.1. *Background*

Let $F_{g,n}$ be the contravariant functor which associates to any k-scheme Ω the set of isomorphism classes of $(\mathbb{Z}/2)^n$-Galois covers $f_\Omega : X_\Omega \to \mathbb{P}^1_\Omega$ where X is a flat Ω-curve whose fibres are smooth projective curves of genus g and where the branch locus B of f_Ω is a simple horizontal divisor. In other words, the branch locus consists of Ω-points of $\mathbb{P}^1_\Omega$ which do not intersect. Since each inertia group is a cyclic group of order 2, the Riemann-Hurwitz formula [4, IV.2.4] implies $g = 2^{n-2}|B| - 2^n + 1$. We assume that $g \geq 1$.

It is well-known that there exists a coarse moduli space $H_{g,n}$ for the functor $F_{g,n}$ which is of finite type over $\mathbb{Z}[1/2]$. (For example, see [10, Chapter 10] when $k = \mathbb{C}$ and [11, Theorem 4] when $p > 2$.) There is a natural morphism $\tau : H_{g,n} \to \mathcal{M}_g$ whose fibres have dimension three. The morphism τ associates to any Ω-point of $H_{g,n}$ the isomorphism class of X_Ω, where $f_\Omega : X_\Omega \to \mathbb{P}^1_\Omega$ is the corresponding cover of Ω-curves. The fibres have dimension three since X_Ω is isotrivial if and only if after an étale base change from Ω to Ω' there is a projective linear transformation ρ such that $\rho f_{\Omega'}$ is constant.

We denote by $\mathcal{H}_{g,n}$ the image $\tau(H_{g,n})$ in $\mathcal{M}_g$. Given a smooth connected k-curve X, then X corresponds to a point of $\mathcal{H}_{g,n}$ if and only if $G \subset \mathrm{Aut}(X)$ with quotient $X/G \simeq \mathbb{P}^1_k$. In particular, $\mathcal{H}_{g,1}$ is simply the locus $\mathcal{H}_g$ of hyperelliptic curves in $\mathcal{M}_g$.

There is also a natural morphism $\beta : H_{g,n} \to \mathbb{P}^{|B|}$ which is proper and étale over the image. The morphism β associates to any Ω-point of $H_{g,n}$ the Ω-point of $\mathbb{P}^{|B|}$ determined by the branch locus of the associated cover. More specifically, β associates to any cover $f_\Omega : X_\Omega \to \mathbb{P}^1_\Omega$ the Ω-point $[c_0 : \ldots : c_{|B|}]$ of $\mathbb{P}^{|B|}$ where c_i are the coefficients of the polynomial whose roots are the branch points of f_Ω. The k-points of the image of β correspond to polynomials with no multiple roots.

It is often more useful to describe the branch locus of f_Ω directly as an Ω-point of $(\mathbb{P}^1)^{|B|}$. This can be done by considering an ordering of the branch points of f_Ω. The branch locus of a cover corresponding to a k-point of $H_{g,n}$ can be any k-point of $(\mathbb{P}^1)^{|B|} - \Delta$ where Δ is the weak diagonal consisting of points having at least two equal coordinates. In particular, for any Ω-point $(b_1, \ldots, b_{2g+2})$ of $(\mathbb{P}^1)^{2g+2} - \Delta$ there is a unique hyperelliptic cover $f_\Omega : X_\Omega \to \mathbb{P}^1_\Omega$ branched at $\{b_1, \ldots, b_{2g+2}\}$ and the curve X_Ω has genus g.

In [3, Section 3.2], we proved several results about the points of $H_{g,n}$. First, by [3, Lemma 3.3], a cover $f : X \to \mathbb{P}^1$ corresponds to a point of $H_{g,n}$ if and only if X has genus g and $f : X \to \mathbb{P}^1$ is isomorphic to the normalized fibre product over $\mathbb{P}^1$ of n smooth hyperelliptic covers $C_i \to \mathbb{P}^1$ whose branch loci B_i satisfy a strong disjointedness condition. In the case $n = 2$, this condition merely says that $B_1 \neq B_2$. Second, suppose $f : X \to \mathbb{P}^1$ is the normalized fibre product over $\mathbb{P}^1$ of n smooth hyperelliptic covers $C_i \to \mathbb{P}^1$ with branch loci B_i. If $H \subset G$ has order 2^{n-1}, in [3, Lemma 3.2] we described the branch locus of the hyperelliptic cover $X/H \to \mathbb{P}^1$ in terms of the branch loci B_i. For example, in the case $n = 2$, the quotient of X by the third involution of $\mathbb{Z}/2 \times \mathbb{Z}/2$ is a hyperelliptic cover of $\mathbb{P}^1$ with branch locus $(B_1 \cup B_2) \backslash (B_1 \cap B_2)$.

We will frequently refer to the following consequence of a theorem of Kani and Rosen. For example, it indicates that if $X \in \mathcal{H}_{g,2}$ then $g(X) = g(X/\alpha_1) + g(X/\alpha_2) + g(X/\alpha_1\alpha_2)$ where α_1, α_2, and $\alpha_1\alpha_2$ denote the three involutions of $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Theorem 1: Suppose X is a smooth projective curve of genus g and $G \subset \mathrm{Aut}(X)$. Suppose $H_i \subset G$ for $1 \leq i \leq w$ are subgroups of G for which $\cup_{i=1}^{w} H_i = G$ and $H_i \cap H_j = \{\mathrm{Id}\}$ for $i \neq j$. Then $(w-1)g(X)+|G|g(X/G) = \sum_{i=1}^{w} |H_i| g(X/H_i)$.

Proof: This follows immediately from [5], in which the authors prove that $(w-1)\,\mathrm{Jac}(X) + |G|\,\mathrm{Jac}(X/G)$ is isogenous to $\sum_{i=1}^{w} |H_i|\,\mathrm{Jac}(X/H_i)$. □

2.2. *Irreducible components of $H_{g,n}$*

This section contains some remarks on the components of $H_{g,n}$ and $\mathcal{H}_{g,n}$. Let $N = 2^n - 1$. We fix an ordering of the N subgroups $H_1, \ldots, H_N$ of G of order 2^{n-1} so that $G = \prod_{i=1}^{n} H_i$.

Proposition 2: *The Hurwitz space $H_{g,n}$ is the disjoint union of irreducible schemes $H_{g,n,\vec{g}}$ indexed by tuples $\vec{g} = (g_1, \ldots, g_N)$ with $g_i \in \mathbb{N}$ and $\Sigma_{i=1}^{N} g_i = g$.*

By [3, Corollary 3.4], for $n \geq 2$ the locus $H_{g,n}$ is empty unless $g \equiv 1 \bmod 2^{n-2}$. If $g \equiv 1 \bmod 2^{n-2}$, then the dimension of each non-empty irreducible component $H_{g,n,\vec{g}}$ is the same number $(g + 2^n - 1)/2^{n-2}$. We note that some of the components $H_{g,n,\vec{g}}$ can be empty.

Proof: A point of $H_{g,n}$ corresponds to a cover $f : X \to \mathbb{P}^1$ along with an isomorphism between G and a subgroup of $\mathrm{Aut}(X)$. The cover f has N hyperelliptic quotients $C_1 \to \mathbb{P}^1, \ldots, C_N \to \mathbb{P}^1$ which are ordered by the fixed choice of ordering of the subgroups of index two of G. Let $\vec{g} = (g_1, \ldots, g_N)$ be the genera of $C_1, \ldots, C_N$. Using [3, Lemma 3.2], one can show that $\Sigma_{i=1}^{N} g_i = g$. For any tuple $\vec{g}$ with $g_i \in \mathbb{N}$ and $\Sigma_{i=1}^{N} g_i = g$, let $H_{g,n,\vec{g}}$ denote the locus of isomorphism classes of covers $f : X \to \mathbb{P}^1$ in $H_{g,n}$ whose hyperelliptic quotients have genera $\vec{g}$. The intersection of two of the strata $H_{g,n,\vec{g}}$ is empty since the tuple corresponding to the cover $f : X \to \mathbb{P}^1$ is well-defined.

To show $H_{g,n,\vec{g}}$ is connected, it is sufficient to show that its fibre over $\mathbb{C}$ is connected by [11, Theorem 4]. By GAGA [9], it is sufficient to work in the topological category. Consider two covers $f : X \to \mathbb{P}^1$ and $f' : X' \to \mathbb{P}^1$ corresponding to points in $H_{g,n,\vec{g}}$. Let B (resp. B') be the branch locus of f (resp. f'). The chosen ordering of the subgroups of index two of G yields a well-defined bijection between the hyperelliptic quotients $C_i \to \mathbb{P}^1$ of f and $C'_i \to \mathbb{P}^1$ of f' for $1 \leq i \leq N$. Since f and f' are in the same locus $H_{g,n,\vec{g}}$, the genus of C_i and the genus of C'_i are equal. This implies that there is a bijection $\iota : B \to B'$ so that b is a branch point of $C_i \to \mathbb{P}^1$ if and only if $\iota(b)$ is a branch point of $C'_i \to \mathbb{P}^1$ for $1 \leq i \leq N$.

There is a morphism $\delta : [0,1] \to (\mathbb{P}^1_{\mathbb{C}})^{|B|} - \Delta$ so that: B is the set of coordinates of $\delta(0)$; B' is the set of the coordinates of $\delta(1)$; $b \in B$ is the jth coordinate of $\delta(0)$ if and only if $\iota(b)$ is the jth coordinate of $\delta(1)$; and the image of δ avoids Δ.

Consider the composition δ_1 of δ with the natural morphism $(\mathbb{P}^1)^{|B|} \to \mathbb{P}^{|B|}$. Since β is étale, δ_1 lifts to a unique morphism $\tilde{\delta} : [0,1] \to (H_{g,n})_{\mathbb{C}}$ with $\tilde{\delta}(0) = f$. In fact, the image of $\tilde{\delta}$ is in $(H_{g,n,\vec{g}})_{\mathbb{C}}$. To see this, note that the lifting $\tilde{\delta}$ yields a deformation of f which must preserve the size of the branch locus and the genus of each hyperelliptic quotient of f.

The fact that $G = \prod_{i=1}^{n} H_i$ implies that f (resp. f') is the normalized fibre product of the hyperelliptic covers $C_i \to \mathbb{P}^1$ (resp. $C'_i \to \mathbb{P}^1$) for $1 \leq i \leq n$, see [3, Lemma 3.3]. It follows that $\tilde{\delta}(1) = f'$ since f' is the unique fibre product of hyperelliptic covers with branch loci B'_i for $1 \leq i \leq n$. Since $(H_{g,n,\vec{g}})_{\mathbb{C}}$ is path connected, it is also connected.

The components $H_{g,n,\vec{g}}$ are smooth since β is an étale cover of a smooth variety. It follows that the components $H_{g,n,\vec{g}}$ are irreducible. □

We will now concentrate on the irreducible components of $\mathcal{H}_{g,n}$ rather than $H_{g,n}$. The dimension of $\mathcal{H}_{g,n}$ equals $\dim(H_{g,n}) - 3$. For example, $\mathcal{H}_{g,2}$ is non-empty of dimension g for all g and $\mathcal{H}_{g,3}$ has dimension $(g+1)/2$ if g is odd and is empty if g is even.

Suppose $\xi \in \mathcal{H}_{g,n}$ and let X be the corresponding curve which admits an action by G. The fibre $\tau^{-1}(\xi) \in H_{g,n}$ corresponds to the set of G-Galois covers $f : X \to \mathbb{P}^1$. There is a natural action of $\mathrm{Aut}(G) = \mathrm{GL}_n(\mathbb{Z}/2)$ on this fibre. The action of $\mathrm{Aut}(G)$ permutes the components $H_{g,n,\vec{g}}$ of $H_{g,n}$ since it permutes the hyperelliptic quotients of X. On the other hand, a permutation of the ordered set of hyperelliptic quotients of X by an element of $\mathrm{GL}_n(\mathbb{Z}/2)$ yields a permutation of this fibre. The irreducible components of $\mathcal{H}_{g,n}$ are indexed by the orbits of the tuples $\vec{g}$ by this action of $\mathrm{Aut}(G)$. In particular, when $n = 2$, $\mathrm{Aut}(G) \simeq S_3$ and thus the components of $\mathcal{H}_{g,2}$ are indexed by unordered triples $\{g_1, g_2, g_3\}$ such that $g_i \in \mathbb{N}$ and $g_1 + g_2 + g_3 = g$.

3. The geometry of $\mathcal{H}_{g,2}$

In this section, we restrict to the case $n = 2$ and investigate the geometry of the locus $\mathcal{H}_{g,2}$. We note that $\mathcal{H}_{g,2}$ is the union of irreducible components

$\mathcal{H}_{g,2,\mathfrak{p}} = \tau(H_{g,2,\vec{g}})$ where $\mathfrak{p}$ ranges over the partitions $\{g_1, g_2, g_3\}$ of g. We assume without loss of generality that $g_1 \leq g_2 \leq g_3$. A priori, these components of $\mathcal{H}_{g,2}$ can intersect since the automorphism group of a curve X may contain several copies of $(\mathbb{Z}/2)^2$. In Section 3.3, we show that this in fact rarely happens.

3.1. *Number of irreducible components*

Lemma 3: *If g is even (resp. odd) then the number of non-empty irreducible components of $\mathcal{H}_{g,2}$ is $[(g+6)^2/48]$ (resp. $[(g+3)^2/48]+\lfloor(g+3)/4\rfloor$).*

Here $[x]$ denotes the integer closest to x and one can check that the fractional parts of the expressions in Lemma 3 do not equal 1/2.

Proof: Let N_g denote the number of non-empty irreducible components $\mathcal{H}_{g,2,\mathfrak{p}}$ of $\mathcal{H}_{g,2}$. These components are indexed by partitions $\mathfrak{p} = \{g_1, g_2, g_3\}$ of g. In the next paragraph, we show that N_g equals the number of such partitions $\mathfrak{p}$ so that $g_3 \leq g_1 + g_2 + 1$.

Suppose $\mathfrak{p}$ is a partition so that $\mathcal{H}_{g,2,\mathfrak{p}}$ is non-empty and let $f : X \to \mathbb{P}^1$ denote a $(\mathbb{Z}/2)^2$-cover corresponding to a point of $\mathcal{H}_{g,2,\mathfrak{p}}$. By [3, Lemma 3.2], the branch loci of the three $\mathbb{Z}/2$-quotients of f satisfy $B_3 = (B_1 \cup B_2)\backslash(B_1 \cap B_2)$. It follows that $|B_3| \leq |B_1| + |B_2|$ and so $g_3 \leq g_1 + g_2 + 1$. Conversely, if $\mathfrak{p} = \{g_1, g_2, g_3\}$ is a partition of g so that $g_1 \leq g_2 \leq g_3$ and $g_3 \leq g_1 + g_2 + 1$, then $\mathcal{H}_{g,2,\mathfrak{p}}$ is non-empty. Namely, $\mathcal{H}_{g,2,\mathfrak{p}}$ contains a point corresponding to the fibre product of two hyperelliptic covers f_1 and f_2 whose branch loci are chosen with the following restrictions: $|B_1| = 2g_1+2$, $|B_2| = 2g_2 + 2$, and $|B_1 \cap B_2| = 2e$ where $e = g_1 + g_2 + 1 - g_3$.

If g is even, then $g_3 \leq g_1 + g_2 + 1$ implies that $g_3 \leq g_1 + g_2$. Thus N_g equals the number of (possibly degenerate) triangles with perimeter g and sides of integer length. By [1], $N_g = [(g + 6)^2/48]$ when g is even.

If g is odd, then $g_3 \neq g_1 + g_2$ and there are two cases to consider. If $g_3 \leq g_1 + g_2 - 1$ then $1 \leq g_1 \leq g_2 \leq g_3$. There is a bijective correspondence between these partitions of g and the partitions $\{a_1, a_2, a_3\}$ of $g - 3$ with $a_1 \leq a_2 \leq a_3$ and $a_3 \leq a_1 + a_2$ (by taking $a_i = g_i - 1$). It follows that these partitions correspond to the number of (possibly degenerate) triangles with sides of integer length and perimeter $g - 3$. By [1], there are $[(g + 3)^2/48]$ such partitions. In the other case, $g_3 = g_1 + g_2 + 1$. The number of these partitions equals the number of pairs $0 \leq g_1 \leq g_2$ so that $g = 2g_1 + 2g_2 + 1$. There are $\lfloor(g + 3)/4\rfloor$ such pairs. □

When $n > 2$, it is more difficult to count the number of nonempty strata, as the interactions between the numbers g_i pose additional restrictions on the partitions.

3.2. *Intersection of Components*

Suppose g is odd. Let $\mathfrak{p}_1 = \{1, (g-1)/2, (g-1)/2\}$ and $\mathfrak{p}'_1 = \{1, (g-3)/2, (g+1)/2\}$. These are the only two partitions $\{1, x, y\}$ of g satisfying the constraint $x \leq y \leq x + 2$. They correspond to two irreducible components of $\mathcal{H}_{g,2}$. In this section we show that these two components intersect. In Section 3.3, we show that the irreducible components corresponding to $\mathfrak{p}_1$ and $\mathfrak{p}'_1$ are the only irreducible components of $\mathcal{H}_{g,2}$ which intersect.

Recall that the dihedral group D_{2c} has presentation $\langle s, r \mid s^2 = r^c = 1,\ srs = r^{-1}\rangle$. Suppose $G = D_{2c} \times \mathbb{Z}/2$ where r and s generate D_{2c} as above and τ is a generator of $\mathbb{Z}/2$.

Lemma 4: *Suppose $c, d \in \mathbb{N}^+$ with c even and d odd. Suppose $G = D_{2c} \times \mathbb{Z}/2$. There exists a $(d+1)$-dimensional family of curves X having an action by G so that $g(X) = 1 + cd$, $g(X/G) = g(X/\langle s, \tau\rangle) = g(X/\langle sr, \tau\rangle) = 0$, and $g(X/s) \neq g(X/sr)$.*

Proof: Let $B = \{0, \infty, 1, \lambda', \lambda_1, \ldots, \lambda_d\}$ be a set of $4 + d$ distinct points in $\mathbb{P}^1$. Let $\gamma_0 = \gamma_\infty = s$, $\gamma'' = sr$, $\gamma' = sr\tau$, and $\gamma_i = \tau$ for $1 \leq i \leq d$. We note that $\gamma_0\gamma_\infty\gamma''\gamma'\Pi_{i=1}^d \gamma_i = 1$ and that these elements generate G. By Riemann's Existence Theorem [10, Theorem 2.13], there exists a $D_{2c} \times \mathbb{Z}/2$ cover $f : X \to \mathbb{P}^1_k$ branched at B so that γ_i is the canonical generator of inertia at a point in the fibre above λ_i (resp. γ_0 above 0, γ_∞ above ∞, γ'' above 1, and γ' above λ'). Above 0 and ∞, the fibre of f consists of $2c$ points; for $1 \leq j \leq c/2$, there are four points in this fibre with inertia group $\langle sr^{2j}\rangle$. Above 1 (resp. λ'), the fibre of f consists of $2c$ points; for $1 \leq j \leq c/2$, there are four points in this fibre with inertia group $\langle sr^{2j-1}\rangle$ (resp. $\langle sr^{2j-1}\tau\rangle$). For $1 \leq i \leq d$, the fibre of f consists of $2c$ points each of which has inertia $\langle\tau\rangle$. By the Riemann-Hurwitz formula [4, IV.2.4], $g(X) = 1 + cd$.

Suppose σ has order 2 in G. Let $F_\sigma = \{P \in X \mid \sigma(P) = P\}$. By the Riemann-Hurwitz formula, $g(X/\sigma) = (g(X)+1)/2 - |F_\sigma|/4$. We calculate:

σ	s	$s\tau$	sr	$sr\tau$	τ
$\lvert F_\sigma\rvert$	8	0	4	4	$2cd$
$g(X/\sigma)$	$(g(X)-3)/2$	$(g(X)+1)/2$	$(g(X)-1)/2$	$(g(X)-1)/2$	1

It follows that $g(X/s) \neq g(X/sr)$. The fact that $g(X) = g(X/s) +$

$g(X/\tau) + g(X/s\tau)$ implies that $g(X/\langle s, \tau\rangle) = 0$, [5]. One shows that $g(X/\langle sr, \tau\rangle) = 0$ similarly. □

We note that the group D_{2c} acts on the quotient X/τ which has genus 1. The element r acts on X/τ via translation by a point of order c while s is the hyperelliptic involution.

Proposition 5: *Let g be odd. The irreducible components $\mathcal{H}_{g,2,\mathfrak{p}_1}$ and $\mathcal{H}_{g,2,\mathfrak{p}'_1}$ intersect.*

Proof: Let $g - 1 = cd$ where d is an odd number and c is a power of 2. Applying Lemma 4, there exists a family of dimension $d+1$ of curves X of genus g whose automorphism group contains two copies of the Klein four group (namely $\langle sr, \tau\rangle$ and $\langle s, \tau\rangle$) so that the two sets of quotients yield the two partitions $\mathfrak{p}_1$ and $\mathfrak{p}'_1$. □

We note that if $g \equiv 3 \bmod 4$, then $\mathcal{H}_{g,2,\mathfrak{p}_1} \cap \mathcal{H}_{g,2,\mathfrak{p}'_1}$ contains a family of dimension $(g+1)/2$ whose generic point has automorphism group $(\mathbb{Z}/2)^3$.

3.3. *Nonintersection of Components*

In this section, we prove the converse of Proposition 5: the irreducible components corresponding to $\mathfrak{p}_1$ and $\mathfrak{p}'_1$ are the only irreducible components of $\mathcal{H}_{g,2}$ which intersect. We use this to count the number of connected components of $\mathcal{H}_{g,2}$.

To begin, we study the intersection of $\mathcal{H}_{g,2}$ with $\mathcal{H}_g$ in order to show that $\mathcal{H}_{g,2}$ is not connected for $g > 3$. Let $\mathfrak{p}_0 = \{0, g/2, g/2\}$ if g is even and $\mathfrak{p}_0 = \{0, (g-1)/2, (g+1)/2\}$ if g is odd. Then $\mathfrak{p}_0$ is the unique partition of g of the form $\{0, x, y\}$ where $x \leq y \leq x+1$. As such, it corresponds to a non-empty irreducible component $\mathcal{H}_{g,2,\mathfrak{p}_0}$ of $\mathcal{H}_{g,2}$.

Proposition 6: *The component $\mathcal{H}_{g,2,\mathfrak{p}_0}$ equals $\mathcal{H}_{g,2} \cap \mathcal{H}_g$. It is a connected component of $\mathcal{H}_{g,2}$ for $g > 3$. The sublocus $\mathcal{H}_{g,2}$ of $\mathcal{M}_g$ is connected if and only if $g \leq 3$.*

Proof: By definition, $\mathcal{H}_{g,2,\mathfrak{p}_0} \subset \mathcal{H}_{g,2} \cap \mathcal{H}_g$. Conversely, suppose X corresponds to a point of $\mathcal{H}_{g,2} \cap \mathcal{H}_g$. Then $\mathrm{Aut}(X)$ contains a hyperelliptic involution σ as well as another involution τ. Also $\langle \sigma, \tau\rangle \simeq (\mathbb{Z}/2)^2$ since σ is in the center of $\mathrm{Aut}(X)$. It follows that $X \to X/\langle \sigma, \tau\rangle$ is a Klein four cover of the projective line with a $\mathbb{Z}/2$-quotient of genus 0. Thus X corresponds to a point of $\mathcal{H}_{g,2,\mathfrak{p}_0}$ and $\mathcal{H}_{g,2,\mathfrak{p}_0} = \mathcal{H}_{g,2} \cap \mathcal{H}_g$.

Next we show that $\mathcal{H}_{g,2,\mathfrak{p}_0}$ is a connected component of $\mathcal{H}_{g,2}$ for $g > 3$ by showing that it does not intersect any other component of $\mathcal{H}_{g,2}$. Suppose X is a curve corresponding to a point of $\mathcal{H}_{g,2,\mathfrak{p}_0} \cap \mathcal{H}_{g,2,\mathfrak{p}}$ for some $\mathfrak{p} \neq \mathfrak{p}_0$. We write $\mathfrak{p} = \{a, b, c\}$ with $0 < a \leq g/3$. Let σ (resp. τ) be the involution of X such that $X/\langle\sigma\rangle$ has genus 0 (resp. such that $X/\langle\tau\rangle$ has genus a). As above, $X \to X/\langle\sigma, \tau\rangle$ is a $(\mathbb{Z}/2)^2$-cover of the projective line. This implies that X also corresponds to a point of $\mathcal{H}_{g,2,\mathfrak{p}'}$ with partition $\mathfrak{p}' = \{0, a, g-a\}$. Then $\mathfrak{p}' = \mathfrak{p}_0$ since $\mathfrak{p}_0$ is the unique partition for $\mathcal{H}_{g,2}$ which contains 0. The conditions $a \in \{(g-1)/2, g/2, (g+1)/2\}$ and $a \leq g/3$ imply that such a curve X can exist only if $g \leq 3$. It follows that the intersection $\mathcal{H}_{g,2,\mathfrak{p}_0} \cap \mathcal{H}_{g,2,\mathfrak{p}}$ is empty for $g \geq 4$ and $\mathfrak{p} \neq \mathfrak{p}_0$.

If $g \leq 2$, then $\mathcal{H}_{g,2,\mathfrak{p}_0}$ is the unique component of $\mathcal{H}_{g,2}$. If $g = 3$, then the two components of $\mathcal{H}_{g,2}$ intersect by Lemma 4. For $g > 3$, $\mathcal{H}_{g,2}$ has at least one pair of non-intersecting components. It follows that $\mathcal{H}_{g,2}$ is connected if and only if $g \leq 3$. □

We now consider the intersection of other components of $\mathcal{H}_{g,2}$.

Hypotheses: Suppose $\mathfrak{p} \neq \tilde{\mathfrak{p}}$ are two partitions of g so that $\mathcal{H}_{g,2,\mathfrak{p}} \cap \mathcal{H}_{g,2,\tilde{\mathfrak{p}}} \neq \emptyset$. Let X be a curve corresponding to a point of the intersection. Let $H_1 \subset \mathrm{Aut}(X)$ and $H_2 \subset \mathrm{Aut}(X)$ be the two Klein four groups which yield the partitions $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$. In particular, $H_1 \neq H_2$ and $g(X/H_1) = g(X/H_2) = 0$. We write $H_1 = \langle\alpha_1, \alpha_2\rangle$ and $H_2 = \langle\sigma_1, \sigma_2\rangle$ where $g(X/\alpha_1) \leq g(X/\alpha_2) \leq g(X/\alpha_1\alpha_2)$ and $g(X/\sigma_1) \leq g(X/\sigma_2) \leq g(X/\sigma_1\sigma_2)$.

Lemma 7: *Under the hypotheses above, either $\alpha_1 = \sigma_1$ or $\alpha_1\sigma_1 = \sigma_1\alpha_1$.*

Proof: By hypothesis, $g(X/\alpha_1) \leq g(X)/3$ and $g(X/\sigma_1) \leq g(X)/3$. In fact, if $g \equiv 0 \bmod 3$ (resp. $g \equiv 1 \bmod 3$), then $g(X/\alpha_1)$ and $g(X/\sigma_1)$ cannot both equal $g/3$ (resp. $(g-1)/3$). The reason is that, if $g \equiv 0 \bmod 3$ (resp. $g \equiv 1 \bmod 3$), there is only one partition whose smallest entry is $g/3$ (resp. $(g-1)/3$) and the partitions $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$ are different by hypothesis. It follows that $g(X/\alpha_1) + g(X/\sigma_1) \leq 2g(X)/3 - 1$. Let m be the order of $\alpha_1\sigma_1$. If $Y = X/\langle\alpha_1, \sigma_1\rangle$, then $g(X) + mg(Y) = g(X/\alpha_1) + g(X/\sigma_1) + g(X/\alpha_1\sigma_1)$, [5]. Thus, $g(X/\alpha_1\sigma_1) \geq g(X) + mg(Y) - (2g(X)/3 - 1) \geq g(X)/3 + 1$. By the Riemann-Hurwitz formula, $g(X/\alpha_1\sigma_1) \leq (g(X)+m-1)/m$. Combining the two inequalities, along with the fact that m must be an integer, yields that $m \leq 2$ which implies that $\alpha_1 = \sigma_1$ or $\alpha_1\sigma_1 = \sigma_1\alpha_1$. □

Let X be a curve with an action by G. If γ_1 and γ_2 are conjugate in G, then the function fields of X/γ_1 and X/γ_2 are conjugate and so $g(X/\gamma_1) = g(X/\gamma_2)$. Recall that if c is odd, then all involutions in D_{2c} are conjugate. If c is even, then sr^i and sr^j are conjugate in D_{2c} if and only if i and j have the same parity.

Lemma 8: *Let X be a curve with an action by $G = D_{2c} \times \mathbb{Z}/2$ where $c \geq 2$ is even. Suppose that $g(X/G) = 0$. Then $g(X/\tau) = g(X/\langle \tau, r\rangle) + g(X/\langle \tau, s\rangle) + g(X/\langle \tau, sr\rangle)$.*

Proof: Consider the action of D_{2c} on X/τ. Any element of D_{2c} is in one of the subgroups $\langle r \rangle$ or $\langle sr^i \rangle$ for some $0 \leq i < c$. The intersection of any two of these subgroups has order 1. Applying Theorem 1 to the curve X/τ and using the fact that $g((X/\tau)/D_{2c}) = 0$, we see that

$$cg(X/\tau) = cg(X/\langle r, \tau \rangle) + \sum_{i=0}^{c-1} 2g(X/\langle \tau, sr^i \rangle)$$

Using the conjugacy relationships described above, this simplifies to $g(X/\tau) = g(X/\langle r, \tau\rangle) + g(X/\langle \tau, s\rangle) + g(X/\langle \tau, sr\rangle)$. □

Theorem 9: The intersection of $\mathcal{H}_{g,2,\mathfrak{p}}$ and $\mathcal{H}_{g,2,\tilde{\mathfrak{p}}}$ is nonempty for $\mathfrak{p} \neq \tilde{\mathfrak{p}}$ if and only if $1 \in \mathfrak{p} \cap \tilde{\mathfrak{p}}$.

Proof: The reverse implication is true by Proposition 5. For the forward implication, suppose that $\mathcal{H}_{g,2,\mathfrak{p}} \cap \mathcal{H}_{g,2,\tilde{\mathfrak{p}}}$ is nonempty for some $\mathfrak{p} \neq \tilde{\mathfrak{p}}$ and let X be a curve corresponding to a point of the intersection. This situation satisfies the hypotheses above. By Lemma 7, either $\alpha_1 = \sigma_1$ or $\alpha_1\sigma_1 = \sigma_1\alpha_1$.

Suppose $\alpha_1 = \sigma_1$. Then $G = H_1H_2$ is isomorphic to $D_{2c} \times \mathbb{Z}/2$ for some c, under the identification $\sigma_2 = s$, $\alpha_2 = sr$, and $\alpha_1 = \tau$. If c is odd, then $g(X/\alpha_2) = g(X/\sigma_2)$ which contradicts the hypothesis that $\mathfrak{p} \neq \tilde{\mathfrak{p}}$; so c is even. Note that by hypothesis $g(X/H_1) = g(X/H_2) = 0$ and so $g(X/G) = 0$. It follows from Lemma 8 that $g(X/\alpha_1) = g(X/\langle \alpha_1, \alpha_2\sigma_2 \rangle)$. This implies that $g(X/\alpha_1) \leq 1$ by the Riemann-Hurwitz formula. By Proposition 6, $g(X/\alpha_1) \neq 0$. Thus $1 \in \mathfrak{p} \cap \tilde{\mathfrak{p}}$.

Now suppose $\alpha_1\sigma_1 = \sigma_1\alpha_1$. Consider the group $G_1 = \langle \sigma_1, \alpha_2, \alpha_1 \rangle$ which is isomorphic to $D_{2c_1} \times \mathbb{Z}/2$ and the group $G_2 = \langle \alpha_1, \sigma_2, \sigma_1 \rangle$ which is isomorphic to $D_{2c_2} \times \mathbb{Z}/2$. Here c_1 (resp. c_2) is the order of $\sigma_1\alpha_2$ (resp. $\sigma_2\alpha_1$). If c_1 is odd, then $g(X/\sigma_1) = g(X/\alpha_2)$. If c_2 is odd, then $g(X/\sigma_2) = g(X/\alpha_1)$. Since $\mathfrak{p} \neq \tilde{\mathfrak{p}}$, it follows that c_1 and c_2 are not both odd.

Without loss of generality, we suppose that $g(X/\sigma_1) \leq g(X/\alpha_1)$. If c_1 is odd, then $g(X/\sigma_1) = g(X/\alpha_2)$. By hypothesis, $g(X/\alpha_2) \geq g(X/\alpha_1)$ which implies $g(\alpha_1) = g(\sigma_1)$ in this case. The implication of this is that (after possibly switching the roles of H_1 and H_2) we can simultaneously suppose that c_1 is even and $g(X/\sigma_1) \leq g(X/\alpha_1)$. Then $g(\sigma_1) \leq (g(X) - 2)/3$ by the hypothesis that $\mathfrak{p} \neq \tilde{\mathfrak{p}}$. Let $\ell = c_1/2$.

Consider the automorphism $r = \alpha_2\sigma_1$. Let H be the subgroup of $\mathrm{Aut}(X)$ generated by involutions α_1, α_2, and r^ℓ. Then $H \simeq (\mathbb{Z}/2)^3$ and $g(X/H) = 0$. It follows from [3, Corollary 3.4] that $g(X)$ is odd. By hypothesis, $g(X) = g(\alpha_1) + g(\alpha_2) + g(\alpha_1\alpha_2)$. The Kani-Rosen theorem [5] applied to the action of H on X implies that $3g(X) = \sum_{\sigma \in H^*} g(X/\sigma)$. After simplifying, we see that

$$2g(X) = g(X/r^\ell) + g(X/r^\ell\alpha_1) + g(X/r^\ell\alpha_2) + g(X/r^\ell\alpha_1\alpha_2).$$

By the Riemann-Hurwitz formula, each of the terms on the righthand side of this equation is at most $(g(X) + 1)/2$. It follows that $g(X/r^\ell\alpha_2) + g(X/r^\ell\alpha_1\alpha_2) \geq g(X) - 1$ and that $g(\alpha_2 r^\ell) \geq (g(X) - 3)/2$.

Suppose ℓ is even. In this case, $r^\ell\alpha_2$ and α_2 are conjugate in D_{2c_1}. Then $g(X) = g(\alpha_1) + g(X/r^\ell\alpha_2) + g(X/r^\ell\alpha_1\alpha_2)$. It follows that $g(\sigma_1) \leq g(\alpha_1) \leq 1$. By Proposition 6, $1 \in \mathfrak{p} \cap \tilde{\mathfrak{p}}$

Suppose ℓ is odd. In this case, σ_1 and $r^\ell\alpha_2$ are conjugate in D_{2c_1}. Thus $(g(X) - 2)/3 \geq g(\sigma_1) \geq (g(X) - 3)/2$. This is only possible if $g(X) \leq 5$. The statement of the theorem is vacuous if $g = 1$ or $g = 3$. If $g = 5$, the only partitions are $\mathfrak{p}_0$, $\mathfrak{p}_1$ and $\mathfrak{p}'_1$ and so $1 \in \mathfrak{p} \cap \tilde{\mathfrak{p}}$ by Proposition 6. □

The following small generalization of Theorem 9 shows that any curve satisfying the hypotheses above is part of the family constructed in Lemma 4.

Corollary 10: *Suppose* $X \in \mathcal{H}_{g,2,\mathfrak{p}} \cap \mathcal{H}_{g,2,\tilde{\mathfrak{p}}}$ *for some* $\mathfrak{p} \neq \tilde{\mathfrak{p}}$ *and let* H_1 *and* H_2 *be the two corresponding Klein four sugroups of* $\mathrm{Aut}(X)$. *Then* $H_1 \cap H_2$ *contains an involution* σ_1, *the subgroup* H_1H_2 *is of the form* $D_{2c} \times \mathbb{Z}/2$ *for some even* c, *and* $g(X/\sigma_1) = 1$.

Proof: By earlier work in this section, the result will follow if $\sigma_1 = \alpha_1$. By Theorem 9, $g(X/\alpha_1) = g(X/\sigma_1) = 1$. Strengthening the equations in the proof of Lemma 7, we see that $(g(X) + m - 1)/m \geq g(X/\alpha_1\sigma_1) \geq g(X) - 2$ where m is the order of $\alpha_1\sigma_1$. If $g(X) > 5$, then this implies $m = 1$ and so $\sigma_1 = \alpha_1$. If $g(X) \leq 5$, the result follows from the approach of [6]. When $g = 3$, $H_1H_2 = (\mathbb{Z}/2)^3$ and when $g = 5$, $H_1H_2 = D_8 \times \mathbb{Z}/2$. □

Corollary 11: *The number of non-empty connected components of* $\mathcal{H}_{g,2}$ *equals* $[(g+6)^2/48]$ *if* g *is even and equals* $[(g+3)^2/48] + \lfloor (g-1)/4 \rfloor)$ *if* g *is odd.*

Proof: Theorem 9 implies that if g is even then no pair of irreducible components of $\mathcal{H}_{g,2}$ intersect and if g is odd then there is exactly one pair of irreducible components which intersect. The proof is then immediate from Lemma 3. □

4. The geometry of $\mathcal{H}_{g,3}$

In this section, we prove that $\mathcal{H}_{g,3}$ is not connected for g odd and $g \geq 5$. Recall that $\mathcal{H}_{g,3}$ is empty if g is even, [3,Corollary 3.4]. Again the method is to show that $\mathcal{H}_{g,3} \cap \mathcal{H}_g$ is a connected component of $\mathcal{H}_{g,3}$. We note that this approach will not work for $n \geq 4$ due to the following lemma.

Lemma 12: *The intersection* $\mathcal{H}_{g,n} \cap \mathcal{H}_g$ *is empty for* $n \geq 4$.

Proof: Suppose X is a curve corresponding to a point of $\mathcal{H}_{g,n} \cap \mathcal{H}_g$. Let σ be a hyperelliptic involution of X. Since σ is in the center of $\mathrm{Aut}(X)$, it follows that there exists $H \simeq (\mathbb{Z}/2\mathbb{Z})^n \subset \mathrm{Aut}(X)$ so that $\sigma \in H$. Thus, $X/\sigma \in \mathcal{H}_{g,n-1}$ and $0 = g(X/\sigma) \equiv 1 \bmod 2^{n-3}$ by [3, Corollary 3.4]. This is a contradiction unless $n \leq 3$. □

The irreducible components of $\mathcal{H}_{g,3}$ correspond to equivalence classes of ordered tuples $(g_1, \ldots, g_7)$ with $g_i \in \mathbb{N}$ and $\sum_{i=1}^{7} g_i$, under a natural action of $\mathrm{Aut}((\mathbb{Z}/2)^3) = \mathrm{GL}_3(\mathbb{Z}/2)$. Without loss of generality, we reorder the subgroups H_i of order 4 in $(\mathbb{Z}/2)^3$ so that $g_1 = g(X/H_1) = \min\{g_i\}_{i=1}^7$ and $g_2 = g(X/H_2) = \min\{g_i\}_{i=2}^7$. Then there is a unique third subgroup H_3 of order 4 in $(\mathbb{Z}/2)^3$ so that $H_1 \cap H_2 \subset H_3$ and we suppose that $g_3 = g(X/H_3)$. Finally, we suppose that $g_4 = \min\{g_i\}_{i=4}^7$. There is a unique representative of each equivalence class satisfying these conditions which we denote by the *partition* $\langle g_1, \ldots, g_7 \rangle$. Let $\mathfrak{p}_0' = \langle 0, 0, 0, j, j, j, j+1 \rangle$ if $g = 4j+1$ and $\mathfrak{p}_0' = \langle 0, 0, 0, j, j+1, j+1, j+1 \rangle$ if $g = 4j+3$.

Lemma 13: *The component* $\mathcal{H}_{g,3,\mathfrak{p}_0'}$ *is non-empty and equals* $\mathcal{H}_{g,3} \cap \mathcal{H}_g$.

Proof: First, note that $\mathcal{H}_{g,3} \cap \mathcal{H}_g$ is non-empty. In particular, consider three hyperelliptic covers with branch loci $B_1 = \{0, \infty\}$, $B_2 = \{0, 1\}$, and B_3 so that $|\cup_{i=1}^3 B_i| = (g+7)/2$. The normalized fibre product of these covers yields a curve in $\mathcal{H}_{g,3} \cap \mathcal{H}_g$. It remains to show that the only partition $\mathfrak{p}$ of g for which $\mathcal{H}_{g,3,\mathfrak{p}}$ contains hyperelliptic curves is $\mathfrak{p}'_0$.

Suppose that X corresponds to a point of $\mathcal{H}_{g,3} \cap \mathcal{H}_g$ and let $\mathfrak{p}$ be its partition. Let σ be such that $g(X/\sigma)$ has genus 0. There exists a subgroup $H \simeq (\mathbb{Z}/2)^3 \subset \mathrm{Aut}(X)$ containing σ. There are three hyperelliptic quotients of $X \to X/H$ dominated by X/σ. Each of these must have genus 0. Thus $\mathfrak{p} = \langle 0, 0, 0, g_4, g_5, g_6, g_7 \rangle$ with $\sum_{i=4}^{7} g_i = g$. Note that $g_i \leq (g+3)/4$ by the Riemann-Hurwitz formula.

Let $Y \to X/H$ be a Klein four quotient of $X \to X/H$. Since $Y \to X/H$ is not disjoint from $X/\sigma \to X/H$, it follows that Y corresponds to a point of $\mathcal{H}_{g(Y),2} \cap \mathcal{H}_{g(Y)}$. By Proposition 6, the three hyperelliptic quotients of $Y \to X/H$ have genera $\{0, x, y\}$ where $x \leq y \leq x+1$. The numerical constraints then imply that $\mathfrak{p} = \mathfrak{p}'_0$. $\square$

Proposition 14: *The locus $\mathcal{H}_{g,3,\mathfrak{p}'_0}$ is a connected component of $\mathcal{H}_{g,3}$. The sublocus $\mathcal{H}_{g,3}$ of $\mathcal{M}_g$ is connected if and only if $g \leq 5$.*

Proof: Suppose X corresponds to a point of $\mathcal{H}_{g,3,\mathfrak{p}'_0}$ and let σ be such that $g(X/\sigma) = 0$. Recall that σ is in the center of $\mathrm{Aut}(X)$. By Lemma 12, $H_{g,4}$ is empty. Thus there does not exist a subgroup $H \simeq (\mathbb{Z}/2)^3 \subset \mathrm{Aut}(X)$ which is disjoint from σ. It follows that X does not correspond to a point of $\mathcal{H}_{g,3,\mathfrak{p}}$ for any $\mathfrak{p} \neq \mathfrak{p}'_0$.

For $g = 1$ and $g = 3$, $\mathfrak{p}'_0$ is the only partition for which $\mathcal{H}_{g,3,\mathfrak{p}}$ is non-empty. It follows that $\mathcal{H}_{1,3}$ and $\mathcal{H}_{3,3}$ are connected.

It now suffices to show for $g \geq 5$ that there exists a partition $\mathfrak{p}$ other than $\mathfrak{p}'_0$ for which $\mathcal{H}_{g,3,\mathfrak{p}}$ is non-empty. Let X' be the normalized fibre product of the three hyperelliptic curves branched at B_1, B_2, and B_3 as follows. If $g = 4j+1$, let $B_1 = \{0, 1\}$, and $B_2 = \{\infty, \mu\}$, and $B_3 = \{\lambda_1, \ldots, \lambda_{2j}, 0, \infty\}$. Then X' corresponds to a point of $\mathcal{H}_{g,3,\mathfrak{p}}$ with $\mathfrak{p} = \langle 0, 0, 1, j, j, j, j \rangle$. If $g = 4j+3$, let $B_1 = \{0, \infty\}, B_2 = \{0, 1, \mu_1, \mu_2\}$, and $B_3 = \{0, 1, \lambda_1, \ldots, \lambda_{2j}\}$. Then X' corresponds to a point of $\mathcal{H}_{g,3,\mathfrak{p}}$ with $\mathfrak{p} = \langle 0, 1, 1, j, j, j, j+1 \rangle$. $\square$

References

1. G. Andrews. A note on partitions and triangles with integer sides. *Amer. Math. Monthly*, 86(6):477–478, 1979.
2. M. Cornalba. On the locus of curves with automorphisms. *Ann. Mat. Pura Appl. (4)*, 149:135–151, 1987.
3. D. Glass and R. Pries. Hyperelliptic curves with prescribed p-torsion. accepted by Manuscripta.

4. R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
5. E. Kani and M. Rosen. Idempotent relations and factors of Jacobians. *Math. Ann.*, 284(2):307–327, 1989.
6. K. Magaard, T. Shaska, S. Shpectorov, and H. Völklein. The locus of curves with prescribed automorphism group. *Sūrikaisekikenkyūsho Kōkyūroku*, (1267):112–141, 2002. Communications in arithmetic fundamental groups (Kyoto, 1999/2001).
7. K. Magaard, S. Shpectorov, and H. Völklein. A GAP package for braid orbit computation and applications. *Experiment. Math.*, 12(4):385–393, 2003.
8. H. Popp. The singularities of the moduli schemes of curves. *J. Number Theory*, 1:90–107, 1969.
9. J.-P. Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble*, 6:1–42, 1955–1956.
10. H. Völklein. *Groups as Galois groups*, volume 53 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1996.
11. S. Wewers. Construction of Hurwitz spaces. Thesis.