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Rigidity, Reduction, and Ramification

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Abstract. In this paper we consider wildly ramified G-Galois covers of curves $f : Y \to \mathbb{P}^1_k$ branched at exactly one point over an algebraically closed field k of characteristic p. For G equal to A_p or $\mathrm{PSL}_2(p)$, we prove Abhyankar's Inertia Conjecture that all possible inertia groups occur over infinity for such covers f. In addition, we prove that the set of conductors that can be realized depends on the group. The method we use is to compute the reduction of Galois covers of $\mathbb{P}^1_{\bar{\mathbb{Q}}}$ branched at 3 points. We observe that the existence of covers with given inertia in characteristic p is closely related to the arithmetic of covers in characteristic zero.

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1. Introduction

Let k be an algebraically closed field of characteristic $p > 2$. Understanding the fundamental group of the affine line over k is a hard problem. By Abhyankar's Conjecture, we know a finite group G occurs as the Galois group of a cover $f : Y \to \mathbb{P}^1_k$ branched only at ∞ if and only if G is a quasi-p group, which means that G is generated by its Sylow p-subgroups, [14]. This determines the set of finite quotients of $\pi_1(\mathbb{A}^1_k)$, but we are far from formulating an analog of Riemann's Existence Theorem. For instance, it is not even known which groups occur as inertia groups above ∞ for such covers f.

Conjecture 1.1 (Inertia Conjecture). Let G be a finite quasi-p group. Let I be a subgroup of G which is an extension of a cyclic group of order prime-to-p by a p-group P. Suppose that the conjugates of P generate G. Then there exists a G-Galois cover $f : Y \to \mathbb{P}^1_k$ branched only at ∞ with inertia group I at some point of $f^{-1}(\infty)$.

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Abhyankar first introduced the Inertia Conjecture, [2, Section 16]. It is classical that the conditions on I are necessary. The only subgroups which were previously known to occur as inertia groups of a G-Galois cover of $\mathbb{P}^1_k$ branched only at ∞ are the Sylow p-subgroups of G, [8].

We show that the Inertia Conjecture is true for A_p and $\mathrm{PSL}_2(p)$, giving the first serious evidence for this conjecture. We give additional results on the question of which filtrations of higher ramification groups occur for such covers f for these groups and for $\mathrm{PSL}_2(\ell)$ where $\ell \neq p$ is a prime such that p exactly divides the order of $\mathrm{PSL}_2(\ell)$.

In the case that p exactly divides the order of G, the inertia group I is an extension of $\mathbb{Z}/n$ by $\mathbb{Z}/p$ with $\gcd(n,p) = 1$ and the filtration is determined by the conductor h. We obtain information on the invariant $\sigma := h/n$ which determines the order of I and the conductor h as long as the prime-to-p part of the center of I is trivial. The following result from Section 3.1 shows that any sufficiently large σ satisfying some obvious necessary conditions occurs.

Theorem 1.2. *Suppose $p \geq 5$. Let G be A_p, $\mathrm{PSL}_2(p)$, or $\mathrm{PSL}_2(\ell)$ (where ℓ is a prime such that p exactly divides $\ell^2 - 1$). There exists an (explicit) constant C, depending on G, such that every $\sigma \geq C$ satisfying the obvious necessary conditions is the ramification invariant of some G-Galois cover $f : Y \to \mathbb{P}^1_k$ branched at exactly one point. Thus the Inertia Conjecture is true for $G = A_p$ and $G = \mathrm{PSL}_2(p)$.*

Our second result, found in Section 3.3, is that the set of conductors that can be realized depends on the group. This result indicates that the question of which filtrations of higher ramification groups occur for such covers may not have an easy to formulate answer. More specifically, despite the fact that the same inertia groups occur for A_p and $\mathrm{PSL}_2(p)$, the set of σ's that occur is different. This is a consequence of the arithmetic of these covers in characteristic 0, namely that $\mathrm{PSL}_2(p)$ has many (almost) rigid triples whereas A_p has very few.

Theorem 1.3. *Suppose that $p \geq 7$. Let $m = (p-1)/2$. There exists an A_p-Galois cover of $\mathbb{P}^1_k$ branched only at ∞ with $\sigma = (p-2)/m$, but there is no $\mathrm{PSL}_2(p)$-Galois cover of $\mathbb{P}^1_k$ branched only at ∞ with $\sigma = (p-2)/m$.*

The main technique we use to construct G-Galois covers of $\mathbb{P}^1_k$ is to reduce covers of $\mathbb{P}^1$ from characteristic zero to characteristic p. This approach, introduced by Raynaud in [14], starts with a G-Galois cover f_K in characteristic zero such that p divides the ramification indices and which thus has bad reduction to characteristic p. Under some conditions on the group, one component of the stable reduction of f_K yields a G-Galois cover of $\mathbb{P}^1_k$ which is wildly ramified above only one point. In Raynaud's work [14] and [15], no statements were made on the inertia groups that occur for the G-Galois covers constructed by reduction. Similarly, there are no statements on the inertia groups in [11] where reduction in equal characteristic is used to prove the existence of a G-Galois cover with $\sigma \leq (2p-1)/(p-1)$ for a more general class of groups G.

In Section 2, we determine the inertia groups which occur in the reduction of some explicit covers f_K of $\mathbb{P}^1_K$. The underlying idea we use to pose restrictions on the inertia groups is a link between arithmetic information (like field of moduli) of covers in characteristic 0 and the ramification of covers in characteristic p; see Section 3.2. We restrict to the case when p exactly divides the order of G and f_K is branched at only three points in order to use the results in [15]. All covers of $\mathbb{P}^1_k$ constructed from the stable reduction of f_K have small σ. We work with covers f_K whose class vector is a rigid triple of conjugacy classes. These covers are often defined over a small number field and thus, roughly speaking, produce covers of $\mathbb{P}^1_k$ with large inertia group.

Even in this restricted setting, we depend on additional information to completely describe the stable reductions in Section 2. For example, when $G = \mathrm{PSL}_2(p)$ we use the reduction of the modular curve $X(2p)$ and when $G = A_p$ we use the existence of subgroups with index p.

In Section 3.1, we use patching in equal characteristic following [11] to obtain existence results for arbitrary large σ and prove the Inertia Conjecture for these groups. In Sections 3.2 and 3.3, we obtain nonexistence results for some small σ using techniques of lifting to characteristic 0. For example, we show that if all G-Galois covers of $\mathbb{P}^1_{\bar{\mathbb{Q}}}$, branched at exactly three points with inertia of order p, are defined over a small number field then there is no G-Galois cover of $\mathbb{P}^1_k$ branched at one point with invariant $\sigma = 2$, Proposition 3.7. We conclude in Section 3.4 with a more general lifting result: every G-Galois cover of $\mathbb{P}^1_k$ branched at exactly one point (such that p strictly divides the order of the inertia group) will occur over a component of the stable reduction of some cover f_K of the projective line in characteristic 0, Proposition 3.11. This proposition simultaneously justifies our approach of reduction in Section 2 and raises an additional question on the relationship between covers in characteristics 0 and p.

2. Stable reduction of Galois covers

The method we use in this paper to construct covers in characteristic p is to reduce covers from characteristic zero to characteristic p. Our method is based on [15]. We start by reviewing some previously known results on the stable reduction of covers branched at three points.

Let G be a finite group such that p strictly divides the order of G. Let $f : Y \to \mathbb{P}^1_k$ be a wildly ramified G-Galois cover of smooth connected k-curves branched at a point ξ. Let I be the inertia group of some point $\eta \in f^{-1}(\xi)$. Let π_η be a uniformizer of Y at η. Then I is an extension of $\mathbb{Z}/n$ by $\mathbb{Z}/p$ with $\gcd(n,p) = 1$. Let n'' be the order of the prime-to-p part of the center of I and $n' = n/n''$. Then $n'|(p-1)$.

Recall from [16, Chapter 4] the *filtration of higher ramification groups* $\{I_i : i \in \mathbb{N}^+\}$ of I: if $g \in I$ then $g \in I_i$ if and only if $g(\pi_\eta) \equiv \pi_\eta \bmod \pi_\eta^{i+1}$. Recall that the *conductor* or *lower jump* h of f at η is the largest integer such that $I_h \neq \{1\}$. The *upper jump*, (in other words, the jump in the filtration of higher inertia groups in the upper numbering) is the *invariant* $\sigma = h/n \in \mathbb{Q}$. Recall that $\gcd(h,p) = 1$ and $\gcd(h,n) = n''$. If $G \neq \mathbb{Z}/p$ and f is branched at exactly one point then $h \geq n + n''$ and thus $\sigma \geq 1 + 1/n'$, [15, Proposition 1.1.6].

2.1. The stable model

Let S be the spectrum of a discrete valuation ring. A *semi-stable* curve over S is a flat proper morphism $X \to S$ whose geometric fibers are reduced connected curves having at most ordinary double points as singularities. A *mark* C on X/S is a closed subscheme of the smooth locus of X which is finite and étale over S. A marked semi-stable curve $(X;C)/S$ is *stably marked* if every geometric fiber of X satisfies the following condition: every irreducible component of genus zero has at least three points which are either singular or lie on the pull-back of the mark C.

Let R_0 be a complete discrete valuation ring whose residue field k is an algebraically closed field of characteristic $p > 2$ and whose fraction field K_0 has characteristic zero. Fix an algebraic closure $\bar{K}$ of K_0. We choose once and for all a compatible system of roots of unity $\zeta_n \in \bar{K}$ for all integers n. Let G be a finite group whose order is strictly divisible by p and $f_{K_0} : Y_{K_0} \to \mathbb{P}^1_{K_0}$ a G-Galois cover, defined over K_0, which is branched only at $x_1 = 0, x_2 = 1, x_3 = \infty$. Write $f_{\bar{K}} := f_{K_0} \otimes_{K_0} \bar{K} : Y_{\bar{K}} \to \mathbb{P}^1_{\bar{K}}$. Let $\underline{C} = (C_1, C_2, C_3)$ be a triple of conjugacy classes of G such that for all i the canonical generator of inertia of a point of $Y_{\bar{K}}$ above x_i (with respect to the fixed roots of unity) is an element of C_i. We call $\underline{C}$ the *class vector* of $f_{\bar{K}}$.

Let K/K_0 be a finite extension such that there exists a K-model $f_K : Y_K \to \mathbb{P}^1_K$ of $f_{\bar{K}}$ with stable reduction. Let R be the ring of integers of K. Recall that $f_{\bar{K}}$ has *stable reduction* over K if the following two conditions hold.

(1) The ramification points of $f_{\bar{K}}$ are rational over K.

(2) The curve Y_K extends to a curve $\mathcal{Y}$ over R such that $\mathcal{Y}$ together with the closure of the ramification points of f_K is a *stably marked curve.*

The action of G on Y_K extends to $\mathcal{Y}$; let $\mathcal{X} := \mathcal{Y}/G$ be its quotient. The curve $\mathcal{X}$ is semistable, [13, Appendix]. The model $f_R : \mathcal{Y} \to \mathcal{X}$ of f_K is called the *stable model.* We write X (resp. Y) for the special fiber of $\mathcal{X}$ (resp. $\mathcal{Y}$). The natural map $f : Y \to X$ is called the *stable reduction* of f_K.

Write $\mathcal{Z} = \mathbb{P}^1_R$ and $Z := \mathcal{Z} \otimes_R k$. There is a natural map $g_R : \mathcal{X} \to \mathcal{Z}$. The map g_R restricts to the identity on a unique irreducible component of X. We denote this component by Z as well and call it the *original component.* All other irreducible components of X are mapped by $g := g_R \otimes k$ to a point of Z. We say that $f_{\bar{K}}$ has *good reduction* if X is equal to Z. This is equivalent to the fact that $f : Y \to X$ is separable, [4, Proposition 1.1.4], since $f_{\bar{K}}$ is branched at three points. If $f_{\bar{K}}$ does not have good reduction, we say that it has *bad reduction.* In particular, if p divides the order of any inertia group then $f_{\overline{K}}$ has bad reduction, [14, Lemma 6.3.3].

Suppose that $f_{\bar{K}}$ has bad reduction. In [20, Section 1.2] it is shown that $f : Y \to X$ has the following structure. This result is only valid for the reduction of covers branched at three points. All irreducible components of X other than the original component intersect Z in a unique point and do not intersect each other. We call these irreducible components the *tails* of X. The three branch points of $f_{\bar{K}}$ specialize to nonsingular points of X. If $x_{i,K}$ is a branch point whose ramification index is prime to p, it specializes to a tail; otherwise it specializes to the original component Z, [4, Lemma 2.1.2.c]. A tail is called *primitive* if one of the branch points specializes to it; otherwise the tail is called *new.* Let $\mathbb{B}_{\mathrm{prim}}$ (resp. $\mathbb{B}_{\mathrm{new}}$) denote the set of primitive (resp. new) tails and $\mathbb{B} = \mathbb{B}_{\mathrm{new}} \cup \mathbb{B}_{\mathrm{prim}}$. For $b \in \mathbb{B}$ we denote the correponding tail of X by X_b and choose a component Y_b of Y above X_b. We denote by ∞_b the (unique) intersection point of X_b with Z. By the choice of our model, at most one branch point of f_K specializes to any tail.

Since $f_{\bar{K}}$ has bad reduction, the restriction of f to Z is inseparable; the inertia group of a component Y_0 of Y above Z has order p. The map $Y_0 \to Z$ factors as $Y_0 \to V \to Z$, where $Y_0 \to V$ is inseparable of degree p and $V \to Z$ is separable of degree prime-to-p. The cover $V \to Z$ is branched only at the points ∞_b for $b \in \mathbb{B}$ and at the specialization of any branch point $x_{i,K}$ for which p divides the ramification index. The ramification index of $x_{i,K} \otimes k$ in f is the same as the ramification index of $x_{i,K}$ in $f_{\bar{K}}$, [14, Lemma 6.3.3]. The restriction of f to a tail of X is separable, [15, Lemma 3.1.2]. The restriction of f to a primitive tail X_b is wildly ramified above ∞_b, tamely ramified above the specialization of the branch point $x_{i,K}$ of f_K and unramified elsewhere. The restriction of f to a new tail X_b is wildly ramified above ∞_b and unramified elsewhere. For each tail X_b, we denote by σ_b the ramification invariant corresponding to the unique wild ramification point of the restriction of f to X_b.

Proposition 2.1. *Let e be the ramification index of p in K_0. Let D be the decomposition group of some component of Y above the original component Z. Let D' be the quotient of D by the prime-to-p part of its center and $n' := |D'|/p$. Then $(p-1)|en'$.*

Proof. This is essentially proved in [15, Proposition 4.2.11 and Corollary 4.2.12] although it is not stated in the same form. For a more general version see [20]. We sketch the argument.

Recall that by assumption f_{K_0} has bad reduction. Therefore f is inseparable over the original component Z. Let η be the generic point of Z. Let S be the complete local ring of $\mathcal{X}$ at η and let S' be the complete local ring of a point of $\mathcal{Y}$ above η. Suppose that D is the Galois group of the Galois extension $\mathrm{Frac}(S')/\mathrm{Frac}(S)$. Then the quotient D' of D by the prime-to-p part of its center is an extension of $\mathbb{Z}/n'$ by $\mathbb{Z}/p$. We conclude as in [15, Corollary 4.2.12] that $(p-1)|en'$. □

2.2. Reduction of a cover with Galois group $\mathrm{PSL}_2(p)$

Let $p \geq 5$ be a prime and $m = (p-1)/2$. In this section we compute the reduction of the covers of $\mathbb{P}^1$ with Galois group $G = \mathrm{PSL}_2(p)$ which are branched at exactly three points of order p. Recall that the order of G is $p(p^2-1)/2$ and that the center of G is trivial. Let

$$P = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

be the standard Sylow p-subgroup of G. The normalizer $N_G(P)$ of P in G consists of the upper triangular matrices and has order pm. Let $W(k)$ be the ring of Witt vectors of the algebraically closed field k of characteristic p. Let $R_0 = W(k)(\sqrt{p})$ (resp. $R_1 = W(k)(\zeta_p)$). For $i = 0, 1$, let K_i be the fraction field of R_i.

The group G has two conjugacy classes of elements of order p, denoted by pA and pB. We make the convention that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in pA.$$

The class vectors $\underline{C}^1 := (pA, pA, pA)$ and $\underline{C}^2 := (pB, pB, pB)$ are both rigid, [18, Lemma 3.27]. (See [17, Section 7.3] for the definition of rigidity.)

This implies that there is a unique G-Galois cover $f^i_{\bar{K}} : Y^i_{\bar{K}} \to \mathbb{P}^1_{\bar{K}}$ with class vector $\underline{C}^i$, [17, Theorem 8.1.1]. Since

$$(pA)^n = pB \quad \text{for} \left(\frac{n}{p}\right) = -1,$$

the class vectors $\underline{C}^1$ and $\underline{C}^2$ are rational over K_0, [17, Section 7.1]. Therefore there exists a model $f^i_{K_0}$ of $f^i_{\bar{K}}$ over K_0. One checks, using again [18, Lemma 3.27], that there are no G-Galois covers of $\mathbb{P}^1_{\bar{K}}$ with class vector (pA, pA, pB) or (pA, pB, pB). The class vectors $\underline{C}^1$ and $\underline{C}^2$ are interchanged by an outer automorphism of G. Therefore it suffices to compute the reduction of the unique cover with class vector $\underline{C}^1$. To ease notation, we drop the index ^{1} from the notation and write $f_{K_0} : Y_{K_0} \to \mathbb{P}^1_{K_0}$ for the unique G-Galois cover with class vector $\underline{C} = (pA, pA, pA)$. For any field L containing K_0 we write $f_L = f_{K_0} \otimes_{K_0} L : Y_L \to \mathbb{P}^1_L$.

We deduce the stable reduction of f_{K_0} from results on the reduction of the modular curve $X(2p)$ due to Deligne–Rapoport [5]. Recall that the modular curve $X(N) \simeq \mathbb{H}^*/\Gamma(N)$ is the quotient of the completed complex upper half-plane by the subgroup $\Gamma(N)$ of $\mathrm{PSL}_2(\mathbb{Z})$ of matrices which are congruent to I mod N. The complement of the cusps in $X(N)$ parameterizes elliptic curves together with an isomorphism $\alpha : (\mathbb{Z}/N)^2 \xrightarrow{\sim} E[N]$ such that

$$\langle \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle = \zeta_N.$$

Here $\langle \cdot, \cdot \rangle$ denotes the Weil pairing and ζ_N is the chosen Nth root of unity. The cusps of $X(N)$ parameterize generalized elliptic curves, [5, Definition II.1.12], with a suitable level structure, [5, Section IV.3.1].

In particular, the complement $Y(2)$ of the cusps in $X(2)$ parameterizes elliptic curves E together with an isomorphism $(\mathbb{Z}/2)^2 \simeq E[2]$. This corresponds to an ordering of $E(\mathbb{C})[2]$. Define a map $Y(2) \to \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ by sending (E, α) to the image of $\alpha(1, 0)$ under the natural map $E \to \mathbb{P}^1_{\mathbb{C}}$ of degree two. This map extends to an isomorphim $\iota : X(2) \to \mathbb{P}^1_{\mathbb{C}}$ which identifies $X(2)$ with the λ-line. The cusps of $X(2)$ correspond to the points $0, 1, \infty \in \mathbb{P}^1_{\mathbb{C}}$. There is a natural map $\pi : X(2p) \to X(2)$ which forgets the level-p structure. The map π is Galois and $\mathrm{Gal}(\pi) \simeq \Gamma(2)/\Gamma(2p) \simeq \mathrm{PSL}_2(p)$.

Lemma 2.2. *The Galois covers $f_{\mathbb{C}}$ and $\iota\pi$ are isomorphic. In particular, there exists an isomorphism $X(2p) \simeq Y_{\mathbb{C}}$ such that the following diagram commutes:*

$$\begin{array}{ccc} X(2p) & \xrightarrow{\sim} & Y_{\mathbb{C}} \\ \downarrow \pi & & \downarrow f_{\mathbb{C}} \\ X(2) & \xrightarrow{\iota} & \mathbb{P}^1_{\mathbb{C}}. \end{array}$$

Proof. Let E be a smooth elliptic curve and $\alpha : E[2] \xrightarrow{\sim} (\mathbb{Z}/2)^2$ an isomorphism as above. The corresponding point of $X(2)$ is unbranched in $X(2p) \to X(2)$ since $\Gamma(2)$ does not contain any non-trivial elements of finite order (compare to [10, Corollary 8.4.5]).

Let (E, α) correspond to a cusp of $X(2)$. The curve E is a generalized elliptic curve which consists of two projective lines meeting in two points; i.e. E is a 2-gon in the terminology of Deligne–Rapoport. Every point in the fiber of π above (E, α) is a $2p$-gon, [5, Section VII.2]. The complete local ring of the corresponding cusp in $X(2p)$ is isomorphic to $\mathrm{Spec}(\mathbb{C}[[q^{1/2p}]])$. By [5, Section VII.2.3], there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{C}[[q^{1/2p}]]) & \longrightarrow & X(2p) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}[[q^{1/2}]]) & \longrightarrow & X(2). \end{array}$$

We conclude that (E, α) is branched in π of order p. Therefore π is a G-Galois cover of $\mathbb{P}^1$ branched at exactly three points of order p. As remarked before, this implies that the class vector of π is either $\underline{C}^1$ or $\underline{C}^2$. Both these triples are rigid and interchanged by an outer automorphism of G. Thus $\iota\pi$ and $f_{\mathbb{C}}$ are isomorphic as Galois covers with group $\mathrm{PSL}_2(p)$. In particular, there exists an isomorphism $X(2p) \simeq Y_{\mathbb{C}}$ as stated in the lemma. □

The modular curve $X(N)$ may be defined over $\mathbb{Q}(\zeta_N)$ and has good reduction to characteristic p for p not dividing N, [10, Theorem 3.7.1]. In particular, $X(2)$ extends to a smooth curve $\mathcal{X}(2) \simeq \mathbb{P}^1_{R_0}$ over R_0. Define $X(2,p) = X(2p)/N_G(P)$; it is a quotient of $\mathbb{H}^*$ by the inverse image of $N_G(P)$ in $\Gamma(2)$. In other words,

$$X(2,p) = \mathbb{H}^*/(\Gamma(2) \cap \Gamma_0(p)),$$

where $\Gamma_0(p)$ is the subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ of upper triangular matrices. Denote the model of $X(2,p)$ over R_1 as defined in [5, Section V.1.14] by $\mathcal{W}$. In [5, Theorem VI.6.9] it is shown that the special fiber W of $\mathcal{W}$ has the following properties.

(a) The curve W is semistable.
(b) It consists of two irreducible components W_1 and W_2 which intersect above the supersingular λ-values.

Here we call $\lambda \in \mathbb{P}^1_k \simeq \mathcal{X}(2) \otimes_{R_1} k$ *supersingular* if the elliptic curve E_λ given by the equation $y^2 = x(x-1)(x-\lambda)$ is supersingular. There is a polynomial $\Phi(\lambda)$ of degree $m = (p-1)/2$ such that the elliptic curve E_λ is supersingular if and only if $\Phi(\lambda) = 0$. It is well known that the polynomial Φ does not have a zero at $\lambda = 0$ and $\lambda = 1$. All zeros of Φ are simple. Therefore there are exactly m supersingular λ-values.

Proposition 2.3 (Reduction of f_{K_0}). *Let $f : Y \to X$ be the stable reduction of f_{K_0}. Let Y_0 be an irreducible component of Y above Z.*

(a) The inertia group of Y_0 has order p; the decomposition group of Y_0 has order pm.
(b) The curve X has no primitive tails.
(c) There are m new tails. For every $b \in \mathbb{B}_{\rm new}$, there is a unique component of Y above X_b and $\sigma_b = (m+1)/m$.

Proof. Note from Section 2.1 that f_{K_0} has bad reduction and the inertia group of Y_0 has order p. We may assume Y_0 to be chosen such that its inertia group is the fixed Sylow p-subgroup P of G. Since the inertia group is a normal subgroup of the decomposition group, we conclude that the decomposition group D of Y_0 is a subgroup of $N_G(P) \simeq \mathbb{Z}/p \rtimes \mathbb{Z}/m$. Since f_{K_0} is defined over K_0 and the ramification index of p in K_0 is two, we conclude from Proposition 2.1 that the order of D is pm. In other words $D = N_G(P)$. This proves (a).

Since the ramification indices of f_{K_0} are equal to p, there are no primitive tails by definition. This proves (b).

We claim that X has at least m new tails. Put $\tilde{\mathcal{W}} = \mathcal{Y}/N_G(P)$. It is a semistable model of $X(2,p)$ since $\tilde{\mathcal{W}} \otimes_{R_1} K_1 = Y_{K_1}/N_G(P) \simeq X(2,p)$. Since $\mathcal{W}$ is the stable model of $X(2,p)$, there is a natural map $\tilde{\mathcal{W}} \to \mathcal{W}$. Let us describe the special fiber of this map. The number of irreducible components of Y above Z is $|G|/mp = p+1$. There is a 1-1 correspondence between these components $Y_0, Y_1, \ldots, Y_p$ and the Sylow p-subgroups of G, given by associating to each component its inertia group. The action of $N_G(P)$ on the irreducible components of Y above Z fixes Y_0; the components $Y_1, \ldots, Y_p$ form one orbit. We conclude that π maps Y_0 to W_1 and $Y_1, \ldots, Y_p$ to W_2, or conversely. Since W_1 and W_2 intersect above the m supersingular λ-values, we conclude that X has at least m singular points. This implies that there are at least m new tails, Section 2.1. It is proved in [15, Section 3.4.2] that

$$\sum_{b \in \mathbb{B}_{\rm new}} (\sigma_b - 1) = 1.$$

Moreover, $\sigma_b - 1 \geq 1/m$, for $b \in \mathbb{B}_{\rm new}$, [15, Proposition 1.1.6]. Therefore, $|\mathbb{B}_{\rm new}| = m$ and $\sigma_b = (m+1)/m$, for every $b \in \mathbb{B}_{\rm new}$.

For each $b \in \mathbb{B}$, the cover $Y_b \to X_b$ is branched at a unique point ∞_b (Section 2.1). By Abhyankar's Conjecture for the affine line [14], the decomposition group of Y_b is a quasi-p group. Since $\sigma_b = (m+1)/m$, the inertia group above ∞_b has order pm. The only subgroup of $\mathrm{PSL}_2(p)$ which is a quasi-p group containing $N_G(P)$ is $\mathrm{PSL}_2(p)$ itself, [9, Theorem 8.2.7]. This proves (c). □

Corollary 2.4. *Let $G = \mathrm{PSL}_2(p)$. For $p \geq 5$ there exists a G-Galois cover of $\mathbb{P}^1_k$ branched at exactly one point with $\sigma = (m+1)/m$.*

Proof. This follows from Proposition 2.3 as the reduction of f has a new tail with $\sigma = (m+1)/m$. □

2.3. Reduction of a cover with Galois group A_p

In this section, we construct a Galois cover of $\mathbb{P}^1_k$ branched at exactly one point with Galois group $G = A_p$. We suppose $p \geq 5$ to avoid trivial examples. Recall that the normalizer $N_G(P)$ of a Sylow p-subgroup P in G is an extension of $\mathbb{Z}/m$ by $\mathbb{Z}/p$ where $m = (p-1)/2$ and the action is faithful. Let $R_0 = W(k)$ and K_0 its fraction field.

The approach we use to construct A_p-Galois covers of the affine line is different from the approach used in Section 2.2. We consider G-Galois covers $f_{K_0} : Y_{K_0} \to \mathbb{P}^1_{K_0}$ branched at three points which factor through a cover $g_{K_0} : \mathbb{P}^1_{K_0} \to \mathbb{P}^1_{K_0}$ of degree p. Our approach is inspired by Zapponi's work on the reduction of degree p covers $\mathbb{P}^1 \to \mathbb{P}^1$, [21]. One can use the idea of the proof of Lemma 2.5 to give a new proof of some of his results.

Let $\underline{C} = (2A, (p-1)A, pA)$ be a triple of conjugacy classes of S_p, where nA denotes the conjugacy class of n-cycles in S_p. This triple is rigid and rational over $\mathbb{Q}$, [17, Section 7.4.1]. Since the center of G is trivial, there exists a unique G-Galois cover $f_{K_0} : Y_{K_0} \to \mathbb{P}^1_{K_0}$ branched at $x_1 = 0, x_2 = 1, x_3 = \infty$ with class vector $\underline{C}$. Choose a subgroup $S_{p-1} \subset G$ and write $Y_{K_0} \to W_{K_0}$ for the S_{p-1} subcover of f_{K_0}. The (non-Galois) cover $g_{K_0} : W_{K_0} \to \mathbb{P}^1_{K_0}$ has degree p and is ramified at one point above x_1 of order two, at one point above x_2 of order $p-1$ and above x_3 of order p. Therefore the genus of W_{K_0} is zero. Note that $Y_{K_0} \to W_{K_0}$ is branched at $p-2$ points of order two and at one point of order $p-1$. Since p divides a ramification index of f_{K_0}, this cover has bad reduction. Let $f : Y \to X$ be the stable reduction of f_{K_0}.

For any $b \in \mathbb{B}$, let G_b be the decomposition group of the component Y_b of Y above the tail X_b and let $H_b = G_b \cap S_{p-1}$. Since $[G : S_{p-1}] = p$ is relatively prime to $[G : G_b]$, we conclude that $[G_b : H_b] = p$. Define $W_b = Y_b/H_b$.

Lemma 2.5. *(a) The curve X has no new tails.*
(b) The curve X has two primitive tails X_1 and X_2, where X_i contains the specialization of the branch point x_i. Moreover,

$$\sigma_1 = \frac{p-2}{p-1} \qquad \text{and} \qquad \sigma_2 = \frac{1}{p-1}.$$

(c) The Galois group of $Y_1 \to X_1$ *is* S_p.

Proof. (a) Suppose $b \in \mathbb{B}_{\text{new}}$. Write pm_b for the order of the inertia group of a point of Y_b above ∞_b. Recall that $p \nmid m_b$. Note that $|H_b| = |G_b|/p$ and that p does not divide the number of elements of the fiber of $Y_b \to X_b$ above ∞_b. We conclude that H_b acts transitively on the fiber of $Y_b \to X_b$ above ∞_b. Therefore $Y_b \to W_b$ is branched at exactly one point; the order of the ramification is m_b. Since W_{K_0} has genus zero the genus of W_b is also zero. We conclude that $Y_b \to W_b$ is a tame cover of $\mathbb{P}^1_k$ branched at exactly one point, which is impossible.

(b) As explained in Section 2.1, the curve X has two primitive tails X_1 and X_2. (Since $x_3 = \infty$ is branched in f_{K_0} of order p, the point x_3 specializes to the original component Z.) The cover $Y_1 \to X_1$ is branched at the specialization of x_1 of order 2. Write $\sigma_1 = h_1/m_1$ (with $\gcd(h_1, m_1) = 1$) for the ramification invariant of ∞_1. The Riemann–Hurwitz formula implies

$$2g(Y_1) - 2 = |G_1|(-2) + |G_1|(1 - \frac{1}{2}) + |G_1|(1 - \frac{1}{pm_1} + \sigma_1(1 - \frac{1}{p})). \quad (2.1)$$

The cover $Y_1 \to W_1$ is branched at $p-2$ points above x_1 of order 2 and at the unique point above ∞_1 of order m_1. Moreover, $g(W_1) = 0$. Therefore

$$\begin{aligned} 2g(Y_1) - 2 &= \frac{|G_1|}{p}(-2) + \frac{|G_1|}{p}(p-2)(1 - \frac{1}{2}) + \frac{|G_1|}{p}(1 - \frac{1}{m_1}) \quad (2.2)\\ &= \frac{|G_1|}{p}(\frac{p}{2} - 2 - \frac{1}{m_1}). \quad (2.3) \end{aligned}$$

Combining (2.1) and (2.3) gives the statement of (b) for $i = 1$. The proof for $i = 2$ is the same.

(c) The Galois group $G_1 = \mathrm{Gal}(Y_1, X_1)$ is the Galois group of the Galois closure of $g_1 : W_1 \to X_1$. Recall that g_1 has degree p and has exactly 2 ramification points: one point with ramification index 2 and one with ramification index p. Therefore G_1 is a subgroup of S_p which contains a transposition and a p-cycle. This implies that $G_1 \simeq S_p$. □

One can show that the decomposition group of Y_2 (in the notation of Lemma 2.5) is an extension of $\mathbb{Z}/(p-1)$ by $\mathbb{Z}/p$. Therefore, we cannot use the tail X_2 to construct A_p-covers of the affine line in characteristic p.

Proposition 2.6. *Let* $G = A_p$. *For* $p \geq 5$ *there exists a* G*-Galois cover* $Y \to \mathbb{P}^1_k$ *branched at exactly one point with* $\sigma = (p-2)/m$.

Proof. We consider the A_p-Galois subcover $\tilde{f}_1 : Y_1 \to \tilde{X}_1 \simeq \mathbb{P}^1_k$ of $f_1 : Y_1 \to X_1 \simeq \mathbb{P}^1_k$. This cover has inertia groups of order pm above ∞ and is unramified elsewhere. Its conductor is the same as the conductor of the S_p-Galois cover, i.e. $h = p - 2$, and thus $\sigma = (p-2)/m$. □

Remark 2.7. (a) The A_p-Galois cover of $\mathbb{P}^1_k$ we constructed in Proposition 2.6 is a special case of some covers considered by Abhyankar in [1]. However, Abhyankar does not mention the ramification invariant. One can extend our method by varying the class vector. In this way one can compute the ramification invariants of all the A_p-Galois covers discovered by Abhyankar.

(b) Let $\phi_{K_0} : Y_{K_0} \to \tilde{X}_{K_0}$ be the A_p-Galois subcover of f_{K_0}. The curve $\tilde{X}_{K_0}$ is a projective line and ϕ_{K_0} has class vector (mA, pA, pB), where mA is the conjugacy class containing the square of a $(p-1)$-cycle and pA and pB are the two conjugacy classes of elements of order p in A_p, [17, Section 4.5]. One easily deduces from Lemma 2.5 that the stable reduction ϕ of ϕ_{K_0} has exactly two tails: one primitive tail with $\sigma = 1/m$ and one new tail with $\sigma = (p-2)/m$. The restriction of ϕ to the new tail is the cover constructed in the proof of Proposition 2.6.

2.4. Reduction of some covers with Galois group $\mathrm{PSL}_2(\ell)$

Let ℓ be a prime different from p such that p strictly divides $\ell^2 - 1$. In this section, we consider covers of the affine line over an algebraically closed field k of characteristic p with Galois group $G = \mathrm{PSL}_2(\ell)$. Note that p strictly divides the order of G which is $\ell(\ell^2-1)/2$. The normalizer $N_G(P)$ of a Sylow p-subgroup P of G is a dihedral group of order $\ell - 1$ or $\ell + 1$, depending on which of the two is divisible by p. Let $n = |N_G(P)|/2p$. Let $R_0 = W(k)(\zeta_p + \zeta_p^{-1})$ and K_0 be the quotient field of R_0. Note that the ramification index e of p in R_0 is $(p-1)/2$.

Proposition 2.8. *Let $G = \mathrm{PSL}_2(\ell)$. There exists a G-Galois cover of $\mathbb{P}^1_k$ branched at exactly one point with $\sigma = 3/2$.*

Proof. The triple $\underline{C} := (pA, pA, \ell A)$ is not rigid in general. Let $\underline{C}'$ be a lift of $\underline{C}$ to a triple of conjugacy classes in $\mathrm{SL}_2(\ell)$. One checks as in [18, Lemma 3.27] that $\underline{C}'$ is a rigid triple in $\mathrm{SL}_2(\ell)$. The conjugacy class ℓA is not rational over $\mathbb{Q}_p$, but becomes rational after an extension of $\mathbb{Q}_p$ that is unramified at p. The conjugacy class pA is also not rational over $\mathbb{Q}_p$. The group G has $(p-1)/2$ conjugacy classes of order p which are rational over $\mathbb{Q}_p(\zeta_p + \zeta_p^{-1})$. Using [18, Theorem 3.25], we conclude that there is some cover $f : Y_{K_0} \to \mathbb{P}^1_{K_0}$ with class vector $\underline{C}$ which is defined over K_0.

The cover f_{K_0} has bad reduction to characteristic p since p divides two of the ramification indices. Denote its stable reduction by $f : Y \to X$. Note that $|\mathbb{B}_{\mathrm{prim}}| = 1$. For any $b \in \mathbb{B}$, the inertia group of $Y_b \to X_b$ above ∞_b is a subgroup of $N_G(P)$ and thus is either cyclic of order pd where d divides n or a dihedral group of order $2p$. Therefore the ramification invariant of the primitive tail X_3 satisfies $\sigma_3 \geq 1/2$. From [15, Proposition 1.1.6] we conclude that $\sigma_b \geq 3/2$ for $b \in \mathbb{B}_{\mathrm{new}}$. The vanishing cycle formula [15, Section 3.4.2] states in this case that

$$\sum_{b\in\mathbb{B}_{\mathrm{new}}} (\sigma_b - 1) + \sigma_3 = 1.$$

Therefore there is at most one new tail.

Let Y_0 be an irreducible component of Y above the original component Z with inertia group P. The decomposition group $D(Y_0)$ of Y_0 is contained in $N_G(P)$ which is a dihedral group. Proposition 2.1 implies that the quotient of $D(Y_0)$ by its prime-to-p center is a dihedral group of order $2p$. Therefore $Y_0 \to Z$ factors through a cyclic cover $V \to Z$ of degree 2. At the branch points of $V \to Z$, the original component Z intersects a tail. Let $b \in \mathbb{B}$ be such that ∞_b is a branch point of $V \to Z$. The inertia group of $Y_b \to X_b$ above ∞_b is a dihedral group of order $2p$. This implies that there are at least two tails whose ramification invariant is not an integer. Since there is only one primitive tail, we conclude that there is a new tail X_4. It has ramification invariant $\sigma_4 = 3/2$. The decomposition group of an irreducible component above X_4 is a subgroup of G which is a quasi-p group containing a dihedral group of order $2p$. The only such group is G, [9, Theorem 8.27]. □

Lemma 2.9. *Let $G = \mathrm{PSL}_2(\ell)$. Suppose that $p \geq 5$ and $\ell \equiv \pm 1 \bmod 8$. There exists a G-Galois cover of $\mathbb{P}^1_k$ branched at exactly one point with $\sigma = 2$.*

Proof. The condition $\ell \equiv \pm 1 \bmod 8$ implies that G contains an element of order 4. Let $\underline{C} = (\ell A, pA, 4A)$. One checks that there exist G-Galois covers over $\bar{\mathbb{Q}}$ with class vector $\underline{C}$. (The triple $\underline{C}$ is *not* rigid; there are two covers with this class vector.) Let $f_{\bar{\mathbb{Q}}} : Y_{\bar{\mathbb{Q}}} \to \mathbb{P}^1_{\bar{\mathbb{Q}}}$ be one of these covers. Let K_0 be a field of definition of $f_{\bar{\mathbb{Q}}}$ containing the fraction field of $W(k)$. Write R_0 for the ring of integers of K_0. Since p divides one of the ramification indices of f, the cover has bad reduction to characteristic p.

The vanishing cycle formula [15, Section 3.4.2] states in this case that

$$\sum_{b\in\mathbb{B}_{\mathrm{new}}} (\sigma_b - 1) + \sigma_1 + \sigma_3 = 1,$$

where σ_1 and σ_3 are the ramification invariants of the two primitive tails. We conclude, as in the proof of Proposition 2.8, that there are no new tails and two primitive tails X_1 and X_3 which both have $\sigma = 1/2$. Since there are no proper subgroups of $\mathrm{PSL}_2(\ell)$ containing elements of order 4 and p, [9, Theorem 8.27], the decomposition group of a component above X_3 is equal to G. Write $f_3 : Y_3 \to X_3$ for the restriction of f to X_3. It follows that f_3 is wildly ramified above ∞_3 and tamely ramified of order 4 above the specialization x_3 of the branch point x_{3,K_0}. We choose an isomorphism $X_3 \simeq \mathbb{P}^1_k$ which sends x_3 to zero and and ∞_1 to ∞. Let $g : \mathbb{P}^1_k = \tilde{X}_3 \to \mathbb{P}^1_k = X_3$ be a 4-cyclic cover branched at 0 and ∞. Consider the pull-back diagram

$$\begin{array}{ccc} \tilde{Y}_3 & \xrightarrow{\tilde{g}} & Y_3 \\ \downarrow \tilde{f}_3 & & \downarrow f_3 \\ \tilde{X}_3 & \xrightarrow{g} & X_3. \end{array}$$

Since $\mathrm{PSL}_2(\ell)$ is simple, the covers f_3 and g are disjoint and $\mathrm{Gal}(\tilde{Y}_3, \tilde{X}_3) \simeq \mathrm{PSL}_2(\ell)$. Abhyankar's Lemma [7, Lemma X.3.6] implies that $\tilde{f}_3$ is unramified above 0 and ramified of order p above ∞. Taking quotients does not change the ramification invariant, [16, Proposition IV.14], therefore $f_3 \circ \tilde{g}$ and f_3 have the same ramification invariant (namely 1/2) above ∞. Note that ∞ is branched in $f_3 \circ \tilde{g}$ of order $4p$ and therefore the corresponding conductor is $4 \cdot 1/2 = 2$. A point above ∞ in $\tilde{f}_3$ also has conductor 2, [16, Proposition IV.2]. The inertia group of such a point in $\tilde{f}_3$ has order p and therefore its ramification invariant is $\sigma = 2/1 = 2$. □

3. Ramification

3.1. Inertia Conjecture

In Section 2, we produced some explicit G-Galois covers $f : Y \to \mathbb{P}^1_k$ branched at exactly one point through reduction. The Inertia Conjecture for $G = \mathrm{PSL}_2(p)$ and $G = A_p$ follows directly by applying Abhyankar's Lemma [7, Lemma X.3.6] to these covers. In this section, we instead prove this result in Corollaries 3.3 and 3.5 from the stronger statement that all inertia groups and all sufficiently large conductors occur for such covers f. In this section we restrict to the case $p \geq 5$.

The proof relies on results from [11] which show how the existence of one or two such covers proves the existence of another with possibly smaller inertia group and larger conductor. The proofs of these results use deformation and formal patching to construct a family of covers of $\mathbb{P}^1$ over a base Ω so that: the normalization of the fiber over one point of Ω

is determined by the original covers; the fiber over the generic geometric point of Ω is a cover with the desired properties.

Proposition 3.1. *Suppose $I \subset G$ is an extension of $\mathbb{Z}/n$ by $\mathbb{Z}/p$ with $\gcd(n,p)=1$. Suppose there exists a G-Galois cover of curves $f_1 : Y \to \mathbb{P}^1_k$ branched at only one point with inertia I and conductor h. For each $d \in \mathbb{N}$ such that $\gcd(d,p)=1$, let $n_d = n/\gcd(n,d)$ and let $h_d = dh/\gcd(n,d)$. Write I_d for the subgroup of I of order pn_d. For each such d, there exists a G-Galois cover of curves $f_d : Y_d \to \mathbb{P}^1_k$ branched at only one point with inertia I_d and conductor h_d.*

Proof. The proof will proceed by induction on d. For $d=1$ the statement is true by hypothesis since $n_1 = n$ and $h_1 = h$. We assume the statement is true for all $d' < d$ such that $\gcd(d',p)=1$. Write $d = d_1 + d_2$ with $1 \le d_i \le d-1$ and $\gcd(d_i,p)=1$. By assumption, there exists a G-Galois cover of curves $f_{d_i} : Y_{d_i} \to \mathbb{P}^1_k$ branched at exactly one point with inertia $I_{d_i} = \mathbb{Z}/p \rtimes \mathbb{Z}/n_{d_i}$ (where $n_{d_i} = n/\gcd(n,d_i)$) and conductor $h_{d_i} = d_i h/\gcd(n,d_i)$.

The covers f_{d_1} and f_{d_2} satisfy the numerical hypotheses of [11, Theorem 2.3.7]. To see this, note that $I_{d_i} \subset I$ with index $r_i = \gcd(n,d_i)$. Letting $e = h_{d_1}r_1 + h_{d_2}r_2 = dh$, and $h' = \gcd(h_{d_1},h_{d_2})$, and $g = \gcd(n,e/h')$, we note that $g = \gcd(n,d)$ and that $\gcd(p,e)=1$. Note that the prime-to-p part of the center of I has order $n' = \gcd(n,h_{d_i})$. Note that $\gcd(h_{d_1}/h', h_{d_2}/h') = 1$ and that $h_{d_1}/h' \equiv \gamma h_{d_2}/h' \bmod n$ for some γ with $\gcd(\gamma,n)=1$.

The conclusion from applying [11, Theorem 2.3.7] to the covers f_{d_1} and f_{d_2} is that there exists a G-Galois cover $f_d : Y_d \to \mathbb{P}^1_k$ branched at exactly one point with inertia $I_d \subset I$ of order $pn/g = pn_d$ and with conductor $e/g = h_d$. This proves the statement for the value d. □

Note that Proposition 3.1 implies the following. Let G be a quasi-p group such that p strictly divides the order of G and the prime-to-p part of the center of $N_G(P)$ is trivial. Then the Inertia Conjecture for G is equivalent to the statement that all but finitely many σ occur for G.

Theorem 3.2. *Let $G = \mathrm{PSL}_2(p)$. Let $m = (p-1)/2$. Let $n|m$. Let a be such that $0 < a \le n$ and $\gcd(a,n)=1$. Let $h \in \mathbb{N}$ be such that $\gcd(h,p)=1$ and $h \equiv a \bmod n$. Assume that $h \ge a(m+1)$. Then there exists a G-Galois cover of curves $f : Y \to \mathbb{P}^1_k$ branched at exactly one point with inertia $I = \mathbb{Z}/p \rtimes \mathbb{Z}/n$ and conductor h.*

Proof. Let $h_* = a(m+1)$. Note that any integer $h \in \mathbb{N}$ such that $h \equiv a \bmod n$ and $h \ge a(m+1)$ is of the form $h_* + in$ for some $i \in \mathbb{N}$. Thus by [11, Theorem 2.2.2], it is sufficient to show that there exists a G-Galois cover of curves $f : Y \to \mathbb{P}^1_k$ branched at exactly one point with inertia $I = \mathbb{Z}/p \rtimes \mathbb{Z}/n$ and conductor h_*.

By Corollary 2.4, there exists a G-Galois cover of curves $f_1 : Y \to \mathbb{P}^1_k$ branched at exactly one point with inertia $I = \mathbb{Z}/p \rtimes \mathbb{Z}/m$ and conductor $h = m + 1$. Let r be such that $m = nr$ and note that $1 \leq ar \leq m$. By Proposition 3.1, for $d = ar$ there exists a G-Galois cover of curves $f : Y \to \mathbb{P}^1_k$ branched at exactly one point with inertia $I = \mathbb{Z}/p \rtimes \mathbb{Z}/n_d$ and conductor h_d. Since $\gcd(m, ar) = r \gcd(n, a) = r$, we see that $n_d = m/r = n$ and $h_d = ar(m + 1)/r = a(m + 1)$. □

Corollary 3.3. *The Inertia Conjecture is true when $G = \mathrm{PSL}_2(p)$. In other words, every subgroup $I = \mathbb{Z}/p \rtimes \mathbb{Z}/n$ of G can be realized as the inertia group of a G-Galois cover of $\mathbb{P}^1_k$ which is branched at exactly one point.*

Proof. This is immediate from Theorem 3.2. □

Theorem 3.4. *Let $G = A_p$. Let $m = (p - 1)/2$ and $n|m$. Let a be such that $0 < a \leq n$ and $\gcd(a, n) = 1$. Let $h \in \mathbb{N}$ be such that $\gcd(h, p) = 1$ and $h \equiv -a \bmod n$. Assume that $h \geq a(p - 2)$. Then there exists a G-Galois cover of curves $f : Y \to \mathbb{P}^1_k$ branched at exactly one point with inertia $I = \mathbb{Z}/p \rtimes \mathbb{Z}/n$ and conductor h.*

Proof. This follows from [11, Theorem 2.2.2], Proposition 2.6, and Proposition 3.1 exactly like the proof of Theorem 3.2. □

Corollary 3.5. *The Inertia Conjecture is true for $G = A_p$. In other words, every subgroup $I = \mathbb{Z}/p \rtimes \mathbb{Z}/n$ of G can be realized as the inertia group of a G-Galois cover of $\mathbb{P}^1_k$ which is branched at exactly one point.*

Proof. This is immediate from Theorem 3.4. □

Theorem 3.6. *Let $G = \mathrm{PSL}_2(\ell)$ with $\ell \neq p$ prime. Suppose that p strictly divides $\ell^2 - 1$. There exists a G-Galois cover of curves $f : Y \to \mathbb{P}^1_k$ branched at exactly one point with inertia I and conductor h in the following cases:*

(a) $I = D_p$ for any h with $\gcd(h, 2p) = 1$ and $h \geq 3$;
(b) $I = \mathbb{Z}/p$ for any h with $\gcd(h, p) = 1$ and $h \geq 3$ if $\ell \equiv \pm 3 \bmod 8$ and $h \geq 2$ if $\ell \equiv \pm 1 \bmod 8$.

Proof. By Proposition 2.8, there exists a G-Galois cover $f : Y \to \mathbb{P}^1_k$ branched at only one point with inertia D_p and conductor 3. Statement (a) follows by applying [11, Theorem 2.2.2] to f.

Applying Abhyankar's Lemma [7, Lemma X.3.6] to f implies that there exists a G-Galois cover of curves $f' : Y \to \mathbb{P}^1_k$ branched at exactly one point with inertia $\mathbb{Z}/p$ and conductor 3. If $\ell \equiv \pm 1 \bmod 8$, then Lemma 2.9 shows that such a cover f' exists with conductor 2. Applying [11, Theorem 2.2.2] to f' realizes all the larger conductors. This proves Statement (b). □

Theorem 3.6 does not prove the Inertia Conjecture for $G = \mathrm{PSL}_2(\ell)$ because of the possibility of inertia groups with non-trivial prime-to-p center. It is not immediately clear how to realize the cyclic subgroups $\mathbb{Z}/pn$ of $\mathrm{PSL}_2(\ell)$ as inertia groups of such covers if $n > 1$.

3.2. The case $\sigma = 2$

The goal of this section is to connect the existence of covers in characteristic p to the arithmetic of covers in characteristic zero. As a first case, we analyze G-Galois covers of $\mathbb{P}^1_k$ branched only at ∞ with inertia of order p and $h = 2$. We show, roughly speaking, that such a cover does not exist if all G-Galois covers of $\mathbb{P}^1$ branched at three points of order p in characteristic zero can be defined over a small number field. The connection comes from the fact that every G-Galois cover of $\mathbb{P}^1_k$ with cyclic inertia groups can be lifted to characteristic zero. As an application of Proposition 3.7, we show that there is no $\mathrm{PSL}_2(p)$-Galois cover $Y \to \mathbb{P}^1_k$ branched only at ∞ with $\sigma = 2$.

Proposition 3.7. *Let G be a quasi-p group such that p strictly divides the order of G. Assume that the normalizer $N_G(P)$ of a Sylow p-subgroup P in G has trivial prime-to-p center. Let $\mathcal{C}_p$ be the set of conjugacy classes of elements of order p in G and let $a = |\mathcal{C}_p|$. Define*

$$b = \max_{\substack{(C_1,C_2,C_3)\\ C_i \in \mathcal{C}_p}} \#\{\textit{covers of } \mathbb{P}^1_{\bar{\mathbb{Q}}} \textit{ with class vector } (C_1, C_2, C_3)\}.$$

If $ab < p-1$ there is no cover $Y \to \mathbb{P}^1_k$ branched at exactly one point with $\sigma = 2$.

Proof. Suppose that $f : Y \to \mathbb{P}^1_k$ is a G-Galois cover branched only at ∞ with $\sigma = 2$. The assumption that $N_G(P)$ has trivial prime-to-p center implies that the inertia group of a point of Y above ∞ is cyclic of order p and that the conductor of such a point is 2. Let $R = W(k)[\zeta_p]$ and K its fraction field. By [6, Theorem 2], there exists a lift $f_R : \mathcal{Y} \to \mathbb{P}^1_R$ of f over R, whose geometric generic fiber $f_{\bar{K}} := f_R \otimes_R \bar{K}$ is a G-Galois cover of smooth curves branched at exactly three points with inertia of order p. Let $\underline{C} = (C_1, C_2, C_3)$ be the class vector of $f_{\bar{K}}$. Then $C_i \in \mathcal{C}_p$.

Note that f_R is not the stable model of f_K; the three branch points of f_K specialize to one point on $\mathbb{P}^1_k$. Let $\tilde{f}_R : \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$ be the stable model of f_K. The relation between f_R and $\tilde{f}_R$ is as follows. The curve $\tilde{X} := \tilde{\mathcal{X}} \otimes_R k$ consists of two irreducible components: the original component Z and a unique tail X. The reason for this is that $\mathcal{Y}$ is smooth and to obtain the stably marked model of $\mathcal{Y}$ we only have to separate the specializations of the branch points. The restriction of $\tilde{f} = \tilde{f}_R \otimes_R k$ to X is the original cover $f : Y \to \mathbb{P}^1_k$. Let Y_0 be an irreducible component of $\tilde{Y} = \tilde{\mathcal{Y}} \otimes_R k$ above Z. Recall that $Y_0 \to Z$ factors as $Y_0 \to W \to Z$, where $Y_0 \to W$ is inseparable of degree p and $W \to Z$ is a tame Galois cover. Since $\tilde{X}$ has a unique tail the cover $W \to Z$ is branched at no more than one point. This implies that the degree of $W \to Z$ is one and that the decomposition group of W has order p. Therefore the integer n' defined in Proposition 2.1 is 1.

Let L be the quotient field of $W(k)$. The assumption on the prime-to-p center of $N_G(P)$ implies that the center of G is trivial. Therefore the cover $f_{\bar{K}}$ has a unique minimal field of definition K_0 containing L, [18, Lemma 3.1.d]. Let $L(\underline{C})$ be the smallest subfield of K_0 containing L over which $\underline{C}$ is rational. Summarizing, we have:

$$K = L(\zeta_p) \supset K_0 \supset L(\underline{C}) \supset L.$$

Note that each of the a conjugacy classes of elements of order p in G contains an element of the Sylow p-subgroup P. This implies that the order of $N_G(P)$ is $p(p-1)/a$. Therefore $g \in P - \{1\}$ is conjugated to g^i if and only if $i \in \mathbb{F}_p^\times$ is a $((p-1)/a)$th root of unity. This implies that $[L(\underline{C}) : L] = a$, [17, Section 7.1].

Note that $K_0/L(\underline{C})$ is Galois. The Galois group $\mathrm{Gal}(K_0, L(\underline{C}))$ acts faithfully on the set of covers with class vector $\underline{C}$. The length of the orbit of the element of this set corresponding to $f_{\bar{K}}$ is at most b. Therefore, $[K_0 : L(\underline{C})] \leq b$. Since $K_0 \subset K = L(\zeta_p)$, we conclude that the ramification index e of p in K_0 is equal to $[K_0 : L(\underline{C})] \cdot a \leq ab$. Proposition 2.1 implies that $en' = e$ is divisible by $p-1$. We conclude that $ab \geq p-1$. This proves the proposition. □

Corollary 3.8. *Suppose $p \geq 5$. Let $G = \mathrm{PSL}_2(p)$. There is no G-Galois cover $f : Y \to \mathbb{P}^1_k$ branched at exactly one point with $\sigma = 2$.*

Proof. We use the same notation as in Proposition 3.7. Recall that the normalizer $N_G(P)$ of a Sylow p-subgroup of G has a trivial prime-to-p center. Since the order of $N_G(P)$ is $p(p-1)/2$, we have that $a = 2$. In Section 2.2 we remarked that $b = 1$. Therefore $a \cdot b = 2 < p - 1$. □

Corollary 3.8 also follows from Proposition 2.3. Here is the idea. Suppose there exists a $\mathrm{PSL}_2(p)$-Galois cover $f : Y \to \mathbb{P}^1_k$ branched at exactly one point with $\sigma = 2$. Let $f_R : \mathcal{Y} \to \mathbb{P}^1_R$ be the lift of f to characteristic zero and $\tilde{f} : \tilde{Y} \to \tilde{X}$ the stable reduction of $f_R \otimes_R K$. Proposition 2.3 implies that $\tilde{X}$ has $m = (p-1)/2$ new tails with $\sigma = (m+1)/m$. But by construction $\tilde{X}$ has a unique new tail with $\sigma = 2$. This gives a contradiction.

Let $\ell \equiv \pm 1 \bmod 8$ be a prime such that p strictly divides $\ell^2 - 1$. Lemma 2.9 states that there exists a $\mathrm{PSL}_2(\ell)$-Galois cover of $\mathbb{P}^1_k$ branched at exactly one point with $h = 2$. In this case $a = (p-1)/2$ and $b = 2$, indicating that the inequality $ab < p - 1$ in Proposition 3.7 is optimal.

One might ask whether the converse of Proposition 3.7 holds, i.e. whether $ab \geq p - 1$ implies that there exists a G-Galois cover of $\mathbb{P}^1_k$ branched at exactly one point with $\sigma = 2$. For example, consider $G = A_p$ for $p \geq 7$. There are many G-Galois covers of $\mathbb{P}^1$ defined over $\bar{\mathbb{Q}}$ branched at three points of order p. Using the computer program GAP, one checks that for $p = 7$ there are 9 covers with class vector $(7A, 7A, 7A)$ and 5 covers with class vector $(7A, 7A, 7B)$. So the number ab in Proposition 3.7 is equal to 18. If $p = 11$, we have $ab = 2 \cdot 14719$. This suggests that there will be (many?) A_p-Galois covers $Y \to \mathbb{P}^1_k$ in characteristic $p \geq 7$ branched at only one point with $\sigma = 2$. Namely, if there is no such A_{11}-Galois cover of $\mathbb{P}^1_k$ all 29,438 A_{11}-Galois covers of $\mathbb{P}^1_{\bar{\mathbb{Q}}}$ with class vector $(11A, 11A, 11A)$ can be defined over an extension K of the quotient field of $W(k)$ of degree strictly less than 5. This seems unlikely. However, we have shown $\sigma = 2$ to occur only for $p = 7$. On one hand, the fact that A_p has few rigid triples suggests that all or most σ will occur. On the other, it makes it hard to show the existence of covers with given small σ with our methods.

3.3. A comparison result

Suppose $p \geq 7$. Let $m = (p-1)/2$ and $R_0 = W(k)[\sqrt{p}]$. We denote the fraction field of R_0 by K_0. In this section, we will use some techniques from [20] to show that there is no $\mathrm{PSL}_2(p)$-Galois cover of $\mathbb{P}^1_k$ branched at exactly one point with $\sigma = (p-2)/m$. The idea is as follows. Let $\phi_{K_0} : Y_{K_0} \to \mathbb{P}^1_{K_0}$ be the A_p-Galois cover with class vector (mA, pA, pB) as in Remark 2.7.b. After a coordinate transformation on $\mathbb{P}^1_{K_0}$, we may suppose the branch locus of ϕ_{K_0} to be $0, 1, \infty$. Recall that the stable reduction $\phi : Y \to X$ of ϕ_{K_0} has a unique new tail. The restriction of ϕ to the new tail is an A_p-Galois cover branched at exactly one point with $\sigma = (p-2)/m$. Suppose there exists a $\mathrm{PSL}_2(p)$-cover $g : W \to \mathbb{P}^1_k$ branched at exactly one point with $\sigma = (p-2)/m$. Then one can construct a $\mathrm{PSL}_2(p)$-equivariant cover f out of ϕ by "replacing" the new tail with

g. The cover f lifts to characteristic zero. The generic fiber of the lift is branched at exactly three points of order m, p, p. We reduce it back to characteristic p and derive a contradiction.

Theorem 3.9. *Let $G = \mathrm{PSL}_2(p)$. Suppose $p \geq 7$. There is no G-Galois cover of $\mathbb{P}^1_k$ branched at exactly one point with $\sigma = (p-2)/m$.*

Proof. Let $H = A_p$ and $\varphi_{K_0} : Y_{K_0} \to \mathbb{P}^1_{K_0}$ be the H-Galois cover with class vector (mA, pA, pB) defined over K_0 as in Remark 2.7.b. Let K be a minimal extension of K_0 such that φ_{K_0} has stable reduction over K. Let $g_1 : Y \to X$ be its stable reduction. We write R for the ring of integers of K. Recall that X has three irreducible components: the original component Z; a primitive tail X_1 and a new tail X_n. Let τ_1 (resp. τ_n) be the intersection point of X_1 (resp. X_n) with Z. The decomposition group of a component of Y above Z has order pm. The decomposition group of a component of Y above X_1 has order pm and $\sigma_1 = 1/m$. The decomposition group of a component Y_n of Y above X_n is the full group H and $\sigma_n = (p-2)/m$.

Claim: The field K contains $K_0(\zeta_p, (\zeta_p - 1)^{1/(p-2)})$.
This is proved in [20]. We outline the proof. Let $\mathcal{Z} = \mathbb{P}^1_{R_0}$ be the stable model of X_{K_0} and write $\mathcal{Y}_0$ for the normalization of $\mathcal{Z}$ in the function field of Y_{K_0}. To find R, we first need to enlarge R_0 so that the special fiber Y_0 of $\mathcal{Y}_0$ is reduced. This implies that R contains ζ_p as in the proof of Proposition 2.1. One checks that the minimal extension of $R_0[\zeta_p]$ over which the new tail Y_2 is defined contains $(\zeta_p - 1)^{1/(p-2)}$. The claim follows from this.

Suppose there exists a G-Galois cover $g : W \to \mathbb{P}^1_k$ branched only at ∞ with $\sigma = (p-2)/m$. Let $\hat{g} : \mathrm{Spec}(\hat{\mathcal{O}}_{W,w}) \to \mathrm{Spec}(\hat{\mathcal{O}}_{\mathbb{P}^1_k,\infty})$ (resp. $\hat{\varphi} : \mathrm{Spec}(\hat{\mathcal{O}}_{Y_n,y}) \to \mathrm{Spec}(\hat{\mathcal{O}}_{X_n,\tau_n}))$ be the local cover, where w (resp. y) is some point in the fiber above ∞ (resp. τ_n). Since $\sigma = (p-2)/m \leq 2$, the moduli space of covers of $\mathrm{Spec}(\hat{\mathcal{O}}_{\mathbb{P}^1_k,\infty})$ with invariant σ has dimension two, [12, Proposition 2.2.6]. Thus there exists an isomorphism $\mathbb{P}^1_k \xrightarrow{\sim} X_n$ which sends ∞ to τ_n and which induces an isomorphism $\hat{g} \simeq \hat{\varphi}$, [12, Lemma 2.3.3].

Let $U = Z \cup X_1$. Let V be a connected component of $\varphi^{-1}(U)$. The decomposition group $D(V)$ is equal to the normalizer of some Sylow p-subgroup in H. Since the normalizers of a Sylow p-subgroup in H and G are isomorphic as abstract groups, we may choose an injection $D(V) \hookrightarrow G$. Let $f : Y' \to X$ be the G-equivariant morphism such that the following conditions hold: the restriction of f to X_n equals g; the restriction of f

to U is $\mathrm{Ind}_{D(V)}^{G}(V \to U)$. As in [15, Proposition 3.2.6] one shows that there exists a stable G-Galois cover $f_R : \mathcal{Y}' \to \mathcal{X}$ whose generic fiber $f_K : Y'_K \to X_K$ is branched at three points of order m, p, p and whose special fiber is f. By [18, Lemma 3.29 and Theorem 3.25], $f_K : Y'_K \to X_K$ descends to K_0. This implies that $\Gamma = \mathrm{Gal}(K/K_0)$ acts on the stable reduction f of f_K. In [20] it is shown that the actions of Γ on X_n induced by f and φ are the same. Let Γ_n be the image of Γ in $\mathrm{Aut}_k(W)$.

Since K_0 is a field of definition of f_K, the action of G on Y'_K can be defined over K_0. This implies that the action of Γ_n commutes with the action of G. Since the center of G is trivial we conclude that $G \cap \Gamma_n = \{1\}$. The claim implies that Γ_n acts on X_n via a cyclic group of order $p-2$. Let Γ'_n be a cyclic subgroup of Γ_n of order $p-2$ which acts faithfully on X_n. Note that Γ'_n acts trivially on the original component and therefore fixes τ_n. We obtain the following diagram:

$$\begin{array}{ccc} W & \stackrel{\Gamma'_n}{\to} & W' \\ \downarrow & & \downarrow \\ X_n & \stackrel{\Gamma'_n}{\to} & X'_n. \end{array}$$

The cover $W' \to X'_n$ is a G-Galois cover branched at exactly two points: above one point the ramification is tame of order $p-2$; above the other point the ramification is wild of order $p(p-1)/2$ with $\sigma = 1/m$. We conclude that G contains an element of order $p-2$. This gives a contradiction. □

Corollary 3.10. *Let $p \geq 7$. Let $\Sigma_{1,p}$ (resp. $\Sigma_{2,p}$) be the set of $\sigma \in \mathbb{Q}$ for which there exists a cover of $\mathbb{P}^1_k$ with Galois group $\mathrm{PSL}_2(p)$ (resp. A_p) branched at exactly one point with ramification invariant σ. Then $\Sigma_{1,p} \neq \Sigma_{2,p}$.*

Proof. This follows immediately from Theorem 3.9 and Proposition 2.6. □

3.4. Lifting to characteristic zero

In this section, we show that every cover $f : Y \to \mathbb{P}^1_k$ branched at exactly one point with inertia of order pn can be produced by reducing a suitable cover g_K from characteristic zero. More precisely, we show that there exists a G-Galois cover g_K of $\mathbb{P}^1_K$ over some local field K of characteristic 0 such that f is the restriction of the stable reduction of g_K to one of the tails. We bound the number of branch points needed for g_K in terms of the conductor of f. This observation justifies the methods in this paper:

it is no restriction to only consider covers in characteristic p which come from the reduction from characteristic zero.

Proposition 3.11. *Let $f : Y \to \mathbb{P}^1_k$ be a G-Galois cover branched at only one point with inertia group I of order pn where $\gcd(n,p) = 1$ and conductor h. Let n'' be the order of the prime-to-p part of the center of I. For some local field K of characteristic zero, there exists a G-Galois cover $g_K : \tilde{Y}_K \to \mathbb{P}^1_K$ with the following properties: g_K is branched at one point of order pn'' and at h/n'' points of order p; there exists a tail W of the stable reduction $g : \tilde{Y} \to X$ of g_K such that $g|_W$ is isomorphic to f.*

Proof. Let $n' = n/n''$. We apply Proposition 3.1 to f for the choice of $d = n' - 1$ and note that $\gcd(d,p) = 1$, $n_d = n/\gcd(n, n'-1)$ and $h_d = h(n' - 1)/\gcd(n, n'-1)$. The conclusion is that there exists a G-Galois cover $f_{n'-1} : Y_{n'-1} \to \mathbb{P}^1_k$ branched at exactly one point with inertia $I_{n'-1}$ of order pn_d and conductor h_d. The covers f and $f_{n'-1}$ satisfy the numerical hypotheses of [11, Theorem 2.3.7]. The conclusion of this theorem is that there exists a G-Galois cover of semistable curves $f' : Y'_{R'} \to X'_{R'}$ over $R' := k[[t]]$ with the following properties: the generic fiber $f' \otimes k((t))$ is a G-Galois cover of smooth curves, branched at exactly one point with inertia I' of order pn'' and conductor h; the base of the special fiber $f' \otimes k : Y' \to X'$ consists of two components X_1 and X_2; after normalization the restriction of f' to X_1 (respectively X_2) is f (respectively $f_{n'-1}$). Recall that the fact that $\gcd(h, n'') = n''$ is equivalent to the fact that n'' is the order of the prime-to-p center of I'. Thus I' is a cyclic group.

By [6, III, 1.3], the cover $f' \otimes k((t))$ lifts to a G-Galois cover $g_K : \tilde{Y}_K \to \mathbb{P}^1_K$ over some local field K of characteristic zero. By [6, II, 6.1], the branch point of f' contributes the following to the branch locus for g_K: one branch point with inertia of order pn'' and h/n'' branch points with inertia of order p. By construction, f' is isomorphic to the restriction of the stable model of g_K over one of the tails. □

In the proofs of Proposition 3.7 and Theorem 3.9 we used ideas similar to Proposition 3.11 to prove non-existence results. Essentially, we showed that every G-Galois cover f of $\mathbb{P}^1_k$ branched at exactly one point with $\sigma = 2$ (resp. $\sigma = 2(p-2)/(p-1)$) occurs over the tail of the stable reduction of a cover g_K of $\mathbb{P}^1_K$ branched at exactly three points. When $G = \mathrm{PSL}_2(p)$, the fact that g_K has only three branch points enabled us to show that the stable reduction of g_K is incompatible with f. One might hope to use this idea to prove more general non-existence results. A problem is that in general g_K has more than three branch points. This makes it harder to compute its stable reduction as the position of the branch points becomes important. Partial results on the reduction of

G-Galois covers branched at four points have been obtained in [4] and [3]. But these results assume that p does not divide the order of the ramification indices and therefore do not apply to the covers obtained in Proposition 3.11.

Question 3.12. Let $f : Y \to \mathbb{P}^1_k$ be a G-Galois cover branched at exactly one point with inertia group I an extension of $\mathbb{Z}/n$ by $\mathbb{Z}/p$ with n prime-to-p and invariant σ. What is the smallest integer r for which there exists a G-Galois cover $g_K : \tilde{Y}_K \to \mathbb{P}^1_K$ branched at exactly r points in characteristic zero such that f is the restriction of the stable reduction of g_K to some new tail?

If $\sigma \leq 2$ and I has no prime-to-p center, one can show that $r = 3$ using techniques similar to those in [19, Section 3.5]. Answering Question 3.12 more generally might give a better insight in the relationship between the existence of covers in characteristic p and zero.

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