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Géometrie algébraique/*Algebraic Geometry*

Conductors of wildly ramified covers, I

Rachel J. Pries

[a] **Department of Mathematics, Columbia University, New York, NY 10027, USA**
E-mail address: pries@math.columbia.edu

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Abstract. Consider a wildly ramified G-Galois cover of curves $\phi : Y \to \mathbb{P}^1_k$ branched at only one point over an algebraically closed field k of characteristic p. For any p-pure group G whose Sylow p-subgroups have order p, I show the existence of such a cover with small conductor. The proof uses an analysis of the semi-stable reduction of families of covers. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Conducteurs des revêtements avec ramification sauvage

Résumé. *Soit k un corps algébriquement clos de caractéristique p. Soit $\phi : Y \to \mathbb{P}^1_k$ un revêtement fini galoisien, de groupe G, ramifié seulement au-dessus d'un point (avec ramification sauvage). Quand G est p-pur et les p-Sylow de G sont d'ordre p, on montre qu'il existe un revêtement de ce type avec un conducteur petit. La démonstration consiste à étudier la réduction semi-stable des familles des revêtements.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Let k be an algebraically closed field of characteristic p. Abhyankar's Conjecture (Raynaud [5]) states that there exists a G-Galois cover $\phi : Y \to \mathbb{P}_k$ branched at only one point if and only if G is a quasi-p group which means that G is generated by p-groups. An open problem is to determine which filtrations of higher ramification groups can be realized for the inertia groups of such a cover ϕ.

Let S be a chosen Sylow p-subgroup of G. In this note, I restrict to the case that S has order p. Under this assumption, any inertia group of ϕ is of the form $I \simeq \mathbb{Z}/p \rtimes \mu_m$ with $\gcd(p, m) = 1$. Furthermore, the filtration of higher ramification groups at a ramification point η is determined by one integer j, namely by the lower jump or conductor; note that $j = \mathrm{val}(g(\pi_\eta) - \pi_\eta) - 1$ where $\mathrm{id} \neq g \in S$ and π_η is a uniformizer at η. Note that $\gcd(p, j) = 1$ and the order n' of the prime-to-p part of the center of I equals $\gcd(j, m)$. When $G \neq \mathbb{Z}/p$, there is a nontrivial lower bound for j. In this case, under an additional hypothesis on G, I show the existence of such a cover ϕ with small conductor, Theorem 3.5.

The main idea of the proof is that it is possible to decrease the ramification data of a given G-Galois cover $\phi : Y \to \mathbb{P}^1_k$. The method is to use [4] to deform the original cover ϕ to a family of covers having a fibre

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ϕ_K with bad reduction. I analyze the special fibre of the semi-stable model of ϕ_K to find new covers of $\mathbb{P}^1_k$ each branched at only one point. Under a condition on G, one of these covers will be connected. Theorem 2.8 compares the ramification information of these covers and of ϕ_K. This is motivated by [5],[6].

Suppose $f : Y \to X$ is a morphism of schemes, ξ is a point of X, and $\eta \in f^{-1}(\xi)$. The *germ* $\hat{X}_\xi$ of X at ξ is the spectrum of the complete local ring of functions of X at ξ and $\hat{f}_\eta : \hat{Y}_\eta \to \hat{X}_\xi$.

2. Degeneration of covers

Let $R \simeq k[[t]]$ where $k = \overline{k}$ has characteristic $p > 2$ and let $K = \mathrm{Frac}(R)$. In this section, all R-curves are proper, normal, reduced and flat over R with smooth and geometrically connected generic fibres. All covers of R-curves are flat and generically separable. We analyze the semi-stable model of the special fibre of a cover ϕ of R-curves with bad reduction. The results follow those of Raynaud [5], [6] where R has unequal characteristic. See also [7].

LEMMA 2.1. – *Suppose that $f : Y \to X$ is a cover of normal curves over R with X_k and Y_k reduced. Let x_R be an R-point of X which specializes to a smooth point x of X_k. Let $y \in f^{-1}(x)$ and suppose $\hat{f}_y$ is étale outside x_R. Let e be the ramification index of $\hat{f}_{y,K}$ over the point $x_K = x_R \times_R K$. If $\gcd(e, p) = 1$ then y is smooth and $\hat{f}_{y,k}$ is tamely ramified at x with ramification index e.*

Proof. – The proof is the same as in unequal characteristic, which was proved in [5, 6.3.2] using Abhyankar's Lemma. See also [7, 1.7] for a proof using Kato's formula [2]. □

LEMMA 2.2. – *Let $f : Y \to X$ be a Galois cover of integral semi-stable R-curves. Let y_K be a rational point of Y_K specializing to a point y of Y_k. Assume $f : Y_K \to X_K$ is étale outside $f(y_K)$. Let η be the generic point of an irreducible component of Y_k which contains y. Then $I(y_K) \subset I(y)$ and $I(\eta)$ is a p-group normal in the inertia group $I(y)$ at y and in the stablilizer $D(\eta)$ of this component.*

Proof. – The proof is the same as the unequal characteristic case in [5, 6.3.3, 6.3.6]. □

LEMMA 2.3. – *Let $f : Y \to X$ be as in Lemma 2.2 with $x \in X_k$ and $y \in f^{-1}(x)$.*

i) Assume $p \neq 2$. Suppose x is a smooth point of X_k. Suppose that f has at most one branch point x_R specializing to x. Then y is a smooth point of Y_k.

ii) Suppose $\hat{f}_{x,K}$ is étale. If x is a node of X_k then y is a node. If $I(\eta_1)$ and $I(\eta_2)$ are the inertia groups of the generic points of the components of $\hat{Y}_y$ containing y then $\langle I(\eta_1), I(\eta_2)\rangle$ is normal in $I(y)$ and contains the Sylow p-subgroup of $I(y)$.

Proof. –

i) (The proof is similar to [7, 1.11]). If y is a node, let I' be the subgroup of $I(y)$ which stabilizes each of the two components passing through y. Since $\hat{f}_y$ is Galois, I' is of index 2 and normal in $I(y)$. Consider the Galois quotient $\hat{f}'_y : \hat{Y}' \to \hat{X}_x$ of $\hat{f}_y$ by I'. Thus $\hat{f}'$ is a Galois cover of degree two from a singular to a smooth germ of a curve. It is generically étale over $\hat{X}_{x,k}$ and the ramification index e of $\hat{f}'_y$ over x_K divides 2. Since $p \neq 2$, this contradicts Lemma 2.1.

ii) See [7, 1.4, 1.9]. Here is the outline: y is a node since Y is semi-stable and the singularity can only worsen. The subgroup $I' = \langle I(\eta_1), I(\eta_2)\rangle$ is normal in $I(y)$. As in part (i), take the quotient of $\hat{f}_y$ by I'. The resulting morphism $\hat{f}'$ is generically étale. Applying a formula of Kato [2] to $\hat{f}'_y$ implies that it is tame and thus prime-to-p. Thus I' contains the Sylow p-subgroup of $I(y)$.

□

Now let $\phi_K : Y_K \to \mathbb{P}^1_K$ be a flat G-Galois cover of proper, smooth, reduced, geometrically connected curves over $\mathrm{Spec}(K)$ with $\mathrm{genus}(Y_K) \geqslant 2$. Let $Y_{0,R}$ be the normalization of $\mathbb{P}^1_R$ in Y_K and let $\phi_{0,R} : Y_{0,R} \to \mathbb{P}^1_R$. Note that $\phi_{0,k}$ can be generically inseparable and $Y_{0,k}$ can be singular.

Here we assume that ϕ_K is étale away from one (necessarily wild) branch point ∞_K.

After a finite extension R' of R, there exists a minimal semi-stable normal curve Y which is a blow-up of $Y_{0,R}$ and has an action of G so that: the quotient map is a G-Galois cover $\phi : Y \to X$; the irreducible components of Y_k are smooth; and the branch points of ϕ specialize in distinct smooth points of X_k. The curve X is semi-stable and normal and X_k is a tree of projective lines. We call $\phi : Y \to X$ the *stable model* of ϕ_K, [5, 6.3]. Let X_{br} be the component of X_k into which ∞_K specializes to a point ∞_k.

DEFINITION 2.4. – *If Y_k is smooth and ϕ_k is generically étale then ϕ_K has good reduction.*

LEMMA 2.5. – *The cover ϕ_K has good reduction if and only if X_k is irreducible.*

Proof. – If ϕ_K has good reduction, then Y_k is connected by Zariski's Theorem and smooth; thus X_k is irreducible since Y_k is. If X_k is irreducible, then it is smooth. Since the branch points of ϕ_K specialize to distinct points of X_k and since $p \neq 2$, Lemma 2.3 (i) indicates that every point y of Y_k is smooth. Since Y_k is smooth and $\mathrm{genus}(Y) \geqslant 2$ the morphism $\phi_k : Y_k \to X_k$ is generically étale; see [6, 2.4.10]. □

DEFINITION 2.6. – *Suppose ϕ_K has bad reduction. An irreducible component C of X_k is terminal if $C \neq X_{br}$ and C intersects the closure of $X_k - C$ in only one point.*

PROPOSITION 2.7. – *Let $\phi : Y \to X$ be the stable model of ϕ_K. If $\phi : Y \to X$ is generically étale over a component C of X_k then C is terminal. Suppose that η is the generic point of a terminal component C of X_k. Then $|I(\eta)| < |S|$, so ϕ is generically étale over C.*

Proof. – This proof is a modification of [5, 6.3.8], [6, 2.4.8], and [6, 3.1.2] to equal characteristic case. The crucial point is that (taking the initial component to be X_{br}) no wild branch point specializes to a component which needs to be contracted in the proof. □

Suppose that ϕ_K does not have good reduction. By Lemma 2.5, Y_k and X_k are singular. Let $U \subset X_k$ be the union of the non-terminal components of the tree X_k. Choose a connected component V of $\phi^{-1}(U)$. With Proposition 2.7 and Lemmas 2.2, 2.3 (ii), one can show that $I \subset D(V) \subset N_G(S)$. Let $\mathbb{B}$ be the set of terminal components of X_k. For $b \in \mathbb{B}$, let P_b be the corresponding terminal component and let ∞_b be the point of intersection of P_b with U. For each $b \in \mathbb{B}$, let $\sigma_b = j_b/m_b$ be the upper jump of the restriction of ϕ to P_b over ∞_b. Let $\sigma = j/m$ be the upper jump of ϕ_K over ∞_K.

THEOREM 2.8. – *(Key Formula):* $\sigma - 1 = \sum_{b \in \mathbb{B}} (\sigma_b - 1)$.

Proof. – The proof parallels that of [6, (3.4.2)(5)] by constructing a $D(V)$-Galois *auxiliary cover* $\psi : Z \to X$ of semi-stable curves which has the same ramification as ϕ but is easier to analyze. The construction of ψ parallels [6, 3.2], using [3] and [1, Theorem 4]. □

3. Decreasing the conductor

Let $\phi : Y \to \mathbb{P}^1_k$ be a G-Galois cover branched at only one point and having inertia $I \simeq \mathbb{Z}/p \rtimes \mu_m$ and conductor j. When $G \neq \mathbb{Z}/p$, there is a small set of values $j_{\min}(I)$, depending only on I, consisting of the minimal possible conductors for ϕ. Let n be such that $m = nn'$ for n' as in Section 1.

DEFINITION 3.1. – *Define* $j_{\min}(I) = \{j_{\min}(I, a) |\ 1 \leqslant a \leqslant n,\ \gcd(a, n) = 1\}$ *where* $j_{\min}(I, a) = 2m + n'$ *if* $a = 1$ *and* $n = p - 1$ *and* $j_{\min}(I, a) = m + an'$ *otherwise.*

The cover ϕ has a non-isotrivial deformation in equal characteristic p if and only if $j \notin j_{\min}(I)$, [4, Theorem 3.1.11]. If $j \notin j_{\min}(I)$ then $\mathrm{genus}(Y_K) \geqslant 2$. Suppose $1 \leqslant a \leqslant n$ and $j \equiv an' \bmod m$. If $G \neq \mathbb{Z}/p$ then $j \geqslant j_{\min}(I, a)$, by [4, Lemma 1.4.3].

DEFINITION 3.2. – *Let $G(S) \subset G$ be the subgroup generated by all proper quasi-p subgroups G' such that $G' \cap S$ is a Sylow p-subgroup of G'. The group G is p-pure if $G(S) \neq G$.*

This condition was introduced in [5]. If G is quasi-p with $|S| = p$, then G is p-pure if and only if G is not generated by all proper quasi-p subgroups $G' \subset G$ such that $S \subset G'$.

PROPOSITION 3.3. – *Let $\phi : Y \to X$ be the stable model of ϕ_K. If G is p-pure and has no (non-trivial) normal p-subgroups, then for some terminal component P_b of X_k, the curve $Y_b = \phi^{-1}(P_b)$ is connected.*

Proof. – The proof is the same as for the unequal characteristic case, [6, 3.1.7]. □

THEOREM 3.4. – *Let G be a finite p-pure quasi-p group whose Sylow p-subgroups have order $p \neq 2$. Suppose there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ branched at only one point with inertia group $I \simeq \mathbb{Z}/p \rtimes \mu_m$ and conductor $j \notin j_{\min}(I)$. Then there exists a G-Galois cover $\phi_b : Y_b \to \mathbb{P}^1_k$ which is branched at only one point with inertia group $I_b \simeq \mathbb{Z}/p \rtimes \mu_{m_b} \subset N_G(S)$ and conductor j_b satisfying $j_b/m_b < j/m$.*

Proof. – By [4, Theorem 3.3.7], for some proper connected variety Ω, there exists a family of G-Galois covers $\phi_\Omega : Y_\Omega \to P_\Omega$ of flat, proper, semi-stable Ω-curves branched at only one Ω-point such that: for some k-point ω, $\phi \simeq \phi_\omega$; and for some K-point of Ω the pullback $\phi_K : Y_K \to \mathbb{P}^1_K$ has bad reduction.

Consider the stable model $\phi : Y \to X$ for ϕ_K. Since ϕ_K has bad reduction there are at least two terminal components of X_k. By Proposition 3.3, the cover is connected over one of the terminal components P_b. By Proposition 2.7, the restriction $\phi_b : Y_b \to P_b \simeq \mathbb{P}^1_k$ is separable. By Lemma 2.1, ϕ_b is branched only at ∞_b since no ramification of ϕ_K specializes to P_b. Over ∞_b, the cover ϕ_b has some inertia group $I_b \simeq \mathbb{Z}/p \rtimes \mu_{m_b} \subset N_G(S)$ and some conductor j_b. By Theorem 2.8, $\sigma_b = j_b/m_b < j/m = \sigma$. □

THEOREM 3.5. – *Let G be a finite p-pure quasi-p group whose Sylow p-subgroups have order $p \neq 2$. For some $I \simeq \mathbb{Z}/p \rtimes \mu_m \subset G$ and some $j \in j_{\min}(I)$, there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ of smooth connected curves branched at only one point over which it has inertia group I and conductor j. In particular,* $\mathrm{genus}(Y) \leqslant 1 + \#G(p-1)/2p$.

Proof. – By Abhyankar's Conjecture [5, 6.5.3], for some I of the form $\mathbb{Z}/p \rtimes \mu_{m'}$ and some j', there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ with group G which is branched at only one point with inertia group I and conductor j'. If $j' \notin j_{\min}(I)$, Theorem 3.4 implies there exists a G-Galois cover $\phi_b : Y_b \to \mathbb{P}^1_k$ which is branched at only one point with inertia group $I_b \simeq \mathbb{Z}/p \rtimes \mu_{m_b} \subset N_G(S)$ and conductor j_b satisfying $j_b/m_b < j'/m'$. We reiterate this process until the inertia group $I_b = \mathbb{Z}/p \rtimes \mu_{m_b}$ and conductor j_b satisfy $j_b/m_b \leqslant 2 + 1/(p-1)$, which implies $j_b \in j_{\min}(I)$. The condition on $\mathrm{genus}(Y)$ follows directly from Definition 3.1 and the Riemann-Hurwitz formula. □

Example 1. – *Let $p = 11$. The simple group $G = M_{11}$ is quasi-11. The only maximal subgroup containing $\mathbb{Z}/11$ is $\mathrm{PSL}_2(11)$, so G is 11-pure and $N_G(S) = \mathbb{Z}/11 \rtimes \mathbb{Z}/5$. By Theorem 3.5, there exists a G-Galois cover $\phi : Y \to \mathbb{P}^1_k$ branched at only one point, either having inertia $\mathbb{Z}/11$ and conductor 2 or inertia $N_G(S)$ and conductor $6 \leqslant j \leqslant 9$.*

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