Pries: 676 Number Theory. 2010. Homework 7.

Units

1. If d is square-free and odd, find the continued fraction of $\sqrt{d^2+1}$. Use this to find the fundamental unit for $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{10})$, and $\mathbb{Q}(\sqrt{26})$.

2. Let W_K be the set of roots of unity in an algebraic number field K. Show W_K is finite and then show it is cyclic of even order.

3. Show that the numbers $(1-\zeta_p^k)/(1-\zeta_p)$ are units of $\mathbb{Z}[\zeta_p]$ when $p \nmid k$.

Ramification

1. Under what conditions on d does the prime 2 ramify, split, or stay inert in $\mathbb{Q}(\sqrt{d})$?

2. Let L be a bi-quadratic field (i.e, $L = \mathbb{Q}(\sqrt{d_1, d_2})$ where $d_1 \neq d_2$ are square-free).

   (i) Find an example of d_1, d_2, p such that (p) is totally ramified in $\mathcal{O}_L$.

   (ii) Find an example of d_1, d_2, p such that (p) is totally split in $\mathcal{O}_L$.

   (iii) Prove that it is not possible for (p) to be inert in $\mathcal{O}_L$.

   (iv) Show that no primes of $\mathbb{Z}$ are inert in $\mathbb{Q}(\zeta_8)$.

3. There is a correspondence between primes of a ring and valuations. For example, the prime p of $\mathbb{Z}$ corresponds to the p-adic absolute value. There is one other valuation on $\mathbb{Z}$, namely the absolute value function $|\cdot|$. This means that $\mathbb{Z}$ has one other "prime", the prime at infinity, which we denote by p_∞. The analogue of the residue field $\mathbb{Z}/p$ is the completion of $\mathbb{Z}$ for $|\cdot|$, namely $\mathbb{R}$.

   A. Under what conditions on d does p_∞ ramify, split, or stay inert in $\mathbb{Q}(\sqrt{d})$?

   B. Describe the factorization of p_∞ in $\mathbb{Z}[2^{1/3}]$.

4. Let $K = \mathbb{C}(x)$ and $L = K[y]/(y^2 - f(x))$. Describe the primes of $\mathbb{C}[x]$ which are ramified in L when $f(x) \in \mathbb{C}[x]$ is a polynomial with no multiple roots.