**667: Commutative Algebra Homework, spring 2008**

# 1 Week 1 - 1/23: Introduction

1. Prove that $\mathbb{Z}[i]$ is a principal ideal domain by showing that it has a Euclidean algorithm. Hint: given $\alpha, \beta \in \mathbb{Z}[i]$, choose $q \in \mathbb{Z}[i]$ so that $\alpha/\beta - q$ has minimal norm. Let $r = \alpha - q\beta$ and show explicitly that $N(r/\beta) < 1$.

2. Prove the following statements about ideals in the ring $\mathbb{Z}[\sqrt{-5}]$.

   (i) $(3) = (3, 1+\sqrt{-5})(3, 1-\sqrt{-5})$. Similarly, factor the ideal (2).

   (ii) $(1+\sqrt{-5}) = (2, 1+\sqrt{-5})(3, 1+\sqrt{-5})$. Similarly, factor the ideal $(1-\sqrt{-5})$.

   (iii) Although $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ does not have unique factorization, show that the ideal (6) has unique factorization into four ideals.

   (iv) Optional: prove that these four ideals are prime.

3. Read Reid chapter 0. Jointly hand in typed list of 10 insightful responses (themes, questions, exercises).

# 2 Week 2 - 1/28: Noetherian rings, graded rings

1. If $R$ is Noetherian, prove that the ring $R[[x]]$ of formal power series is Noetherian.

2. Let $R = k[x, y]$, which is Noetherian. Prove that the subring $S = k[x, x^2y, x^3y^2, \ldots, x^iy^{i-1}, \ldots]$ is not Noetherian.

3. Let $R \subset S$ be a ring extension. Recall that $s \in S$ is *integral* over $R$ if $s$ is the root of a monic polynomial $f(x) \in R[x]$. Prove that $R[s]$ is a finitely generated $R$-module if and only if $s$ is integral over $R$.

4. An element $m$ of an $R$-module $M$ is a *torsion element* if $rm = 0$ for some $r \in R$ with $r \neq 0$. Let $\mathrm{Tor}(M)$ be the set of all torsion elements of $M$.

   (i) If $R$ is an integral domain, prove that $\mathrm{Tor}(M)$ is a submodule of $M$.

   (ii) Give an example of $R$ and $M$ such that $\mathrm{Tor}(M)$ is not a submodule.

   (iii) If $R$ has zero divisors and $M \neq \{0\}$, show that $\mathrm{Tor}(M) \neq 0$.

5. Let $R$ be a UFD and let $x, y \in R$ be two elements with no common factor. Let $I = (x, y)$. Define $\phi : R^2 \to I$ by $\phi(a, b) = ax + by$ and define $\alpha : R \to R^2$ by $\alpha(r) = (-ry, rx)$. Prove that this gives an exact sequence:

$$0 \to R \xrightarrow{\alpha} R^2 \xrightarrow{\phi} I \to 0.$$

6. If $R$ is a graded ring and $I$ is a homogenous ideal, follow this proof that $R/I$ is a graded ring.

   (i) Check that $R_iI_j \subset I_{i+j}$.

   (ii) Check that the multiplication $(R_i/I_i)(R_j/I_j) \subset R_{i+j}/I_{i+j}$. is well-defined.

   (iii) Briefly explain why $S = \oplus_{i=0}^{\infty}(R_i/I_i)$ is a graded ring.

   (iv) Prove that the map $R = \oplus_{i=0}^{\infty}R_i \xrightarrow{\tau} S$ given by $\tau((r_i)_{i=0}^{\infty}) = (r_i \bmod I_i)_{i=0}^{\infty}$ is surjective with kernel $I = \oplus_{i=0}^{\infty}I_i$.

## 3 Week 3 - 2/4 Properties of ideals:

1. Prove that $\mathrm{SL}_r(k)$ is an affine variety in $\mathbb{A}_k^{r^2}$.

2. Let $k$ be a field (not necessarily algebraically closed). For each non-constant $f \in k[x]$, describe $V(f)$ in terms of the unique factorization of $f$ in $k[x]$. Then use this to describe $I(V(f))$. When is $I(V(f)) = (f)$? Prove your answer.

3. (i) If $I \subset J$, prove $\mathrm{Rad}(I) \subset \mathrm{Rad}(J)$.

   (ii) Prove that $\mathrm{Rad}(\mathrm{Rad}(I)) = \mathrm{Rad}(I)$.

4. Let $I = (xy, (x-y)z)$ and $J = (xy, xz, yz)$. Complete two proofs that $J = \mathrm{Rad}(I)$.

   (i) (For both proofs) show that $\mathrm{Rad}(J) = J$. Hint: consider the nilradical of $R/J$.

   (ii) (For first proof) show that $I \subset J \subset \mathrm{Rad}(I)$ and then conclude.

   (iii) (For second proof) show that $V(I) = V(J)$ and use the Nullstellensatz.

5. Let $W = V(I)$ be an affine variety in $\mathbb{A}_k^n$. Let $f \in k[W]$. The *graph* of $f$ is the following subset of $\mathbb{A}_k^{n+1}$:

$$W_f = \{(a_1, \ldots, a_n, f(a_1, \ldots, a_n)) \mid (a_1, \ldots, a_n) \in W\}.$$

   (i) Prove that $W_f$ is an affine variety.

   (ii) Prove that $k[W_f] \simeq k[W]$. This indicates that $W$ and $W_f$ are isomorphic. Interpret this isomorphism geometrically.

   (iii) If $W$ is irreducible, prove that $W_f$ is irreducible.

6. Prove that a linear variety is irreducible. Hint: if $L_1, \ldots, L_r$ are linear polynomials in $k[x_1, \ldots, x_n]$, prove that $(L_1, \ldots, L_r)$ is prime.

## 4 Week 4 - 2/11: Affine algebraic varieties

1. Suppose $f_1$ and $f_2$ are orthogonal idempotents of $k[V]$. For $i = 1, 2$, let $J_i = k[V] \cdot f_i$. These are non-zero ideals of $k[V]$. Consider the map $\phi : k[V] \to J_1 \oplus J_2$ where $\phi(f) = (ff_1, ff_2)$. Consider the map $\psi : J_1 \oplus J_2 \to k[V]$ where $\psi(h_1 f_1, h_2 f_2) = h_1 f_1 + h_2 f_2$.

   (i) Prove that $\phi$ and $\psi$ are $k[V]$-module homomorphisms.

   (ii) Prove that $\phi \cdot \psi = \mathrm{id}$ and $\psi \cdot \phi = \mathrm{id}$.

2. Finish the proof of the Proposition about connectedness. Recall that $k[V] = J_1 \oplus J_2$ where $J_1, J_2$ are non-zero ideals of $k[V]$. For $i = 1, 2$, let $I_i$ be the ideal of $R$ which is the inverse image of $J_i$ under the morphism $R \to k[V]$ and let $V_i = V(I_i)$. We proved that $V_1 \cap V_2 = \emptyset$. Prove that $V = V_1 \cup V_2$ (both inclusions).

3. Let $f_1(x, y) = x^2 + y^2 - 1$ and $f_2(x, y) = y - 2 - x^2$. Let $V = V((f_1 \cdot f_2))$.

   (i) Is $V$ irreducible? Is it connected?

   (ii) Graph $V$ in $\mathbb{R}^2$ and explain why the picture is misleading.

   (iii) Let $x_1 = ix$ where $i = \sqrt{-1}$. For $j = 1, 2$, find $g_j(x_1, y) \in k[x_1, y]$ s.t. $g_j(x_1, y) = 0$ iff $f_j(x, y) = 0$. Plot $V' = V((g_1 \cdot g_2))$ in $\mathbb{R}^2$ and explain what it signifies.

4. Let $f(x, y) = y^2 - x^3 + x$.

   (i) Prove that $f$ is irreducible in $k[x, y]$ for any field $k$.

   (ii) Prove that $V((f))$ is irreducible.

   (ii) Graph $f(x, y) = 0$ in $\mathbb{R}^2$ and explain why the picture is misleading.

## 5 Week 5 - 2/18: Morphisms/integral closure

1. Consider $\phi : \mathbb{A}^2 \to \mathbb{A}^3$ where $\phi((s, t)) = (st, s + t, s^2 t)$. Find the ideal $I$ of $k[a, b, c]$ of functions that vanish on $\mathrm{Im}(\phi)$. Show that the $k$-points of $V(I)$ and $\mathrm{Im}(\phi)$ are the same.

2. Consider $\phi : \mathbb{A}^2 \to \mathbb{A}^3$ where $\phi((s, t)) = (st, s, t^2)$. Find the ideal $I$ of $k[a, b, c]$ of functions that vanish on $\mathrm{Im}(\phi)$. Show that the $k$-points of $V(I)$ and $\mathrm{Im}(\phi)$ are different for some fields $k$.

3. Let $I = (x^2, xy, xz, yz)$ in $k[x, y, z]$.

   (i) Prove that $I = (x, y) \cap (x, z) \cap (x, y, z)^2$.

   (ii) Find the isolated and embedded primes of $I$.

   (iii) Draw a picture of $V(I)$ and mark the embedded components.

   (iv) Find $\mathrm{rad}(I)$

4. Let $\zeta_p = e^{2\pi i/p}$ be a primitive $p$th root of unity where $p$ is prime. Prove that $\mathbb{Z}[\zeta_p]$ is contained in the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\zeta_p)$. In fact, it equals the integral closure but this is much harder to prove.

5. Let $R = k[x, y]/(x^2 - y^3)$.

   (i) Find an element $z$ of $\mathrm{Frac}(R)$ which is integral over $R$.

   (ii) Prove that $k[z]$ is the integral closure of $R$.

   (iii) How would the answer change for $R = k[x, y]/(x^i - y^j)$ when $i$ and $j$ are relatively prime positive integers.

6. Let $R = k[x, y]/(y^2 - x^3 - x^2)$.

   (i) Find an element $z$ of $\mathrm{Frac}(R)$ which is integral over $R$. (As in the previous problem $k[z]$ is the integral closure of $R$.)

   (ii) Explicitly describe the ring homomorphism $R \to k[z]$.

   (iii) Explicitly describe the morphism $\phi : \mathbb{A} \to V(y^2 - x^3 - x^2)$. What is this morphism doing geometrically?

## 6 Week 6 - 2/25: Normalization

1. Let $K = \mathbb{Q}(a^{1/3})$ where $a$ is a cube-free integer. A basis for $K$ over $\mathbb{Q}$ is $S = \{1, a^{1/3}, a^{2/3}\}$. Let $b = (1 + a^{1/3} + a^{2/3})/3$.

   (i) In terms of the basis $S$, find the matrix for the linear transformation $M_b : K \to K$ where $M_b(z) = bz$.

(ii) Under what condition(s) on $a$, is $b$ integral over $\mathbb{Z}$?

2. Suppose $1 \in R \subset S$, that $S$ is an integral domain, and that $S$ is integral over $R$. If $R$ is a field, prove that $S$ is a field. (We did the converse in class).

3. Let $f$ be a non-constant polynomial in $A[x_m]$ where $A = k[x_1, \dots, x_{m-1}]$. If $k$ is infinite, prove that there exist $a_1, \dots, a_{m-1} \in k$ so that after substituting $x_i = X_i + a_i x_m$ the polynomial $f \in k[X_1, \dots, X_{m-1}, x_m]$ has leading term of the form $a x_m^e$ for some $0 \neq a \in k$. Hint: work with $f_d$, the homogenous part of $f$ of maximum degree $d$.

4. Dummit/Foote 15.3 #6 (a,b,e).

5. Dummit/Foote 15.4 #18.

6. Let $P$ be a prime ideal of an integral domain $R$. Let $D = R - P$ and let $R_P = D^{-1}R$. Prove that the fields $\mathrm{Frac}(R/P)$ and $R_P/PR_P$ are isomorphic.

# 7 Week 7 - 3/3: Localization, tensor product

1. Let $R = k[x, y]$ and let $M = (x, y)$.

   (i) Describe the functions in $R_M$. Describe the maximal ideal of $R_M$.

   (ii) Show there is a bijection between irreducible curves $C$ in $\mathbb{A}^2$ such that $(0, 0) \in C$ and prime ideals of $R_M$.

2. Let $D$ be a multiplicatively closed subset of $R$ and let $R \subset S$. Let $\mathcal{O}_R$ be the integral closure of $R$ in $S$. Let $\mathcal{O}$ be the integral closure of $D^{-1}R$ in $D^{-1}S$.

   (i) Prove that $D^{-1}\mathcal{O}_R \subset \mathcal{O}$.

   (ii) Prove that $\mathcal{O} \subset D^{-1}\mathcal{O}_R$. Hint: If $s/d \in D^{-1}S$ is integral over $D^{-1}R$, show that there exists $\delta \in D$ such that $\delta s \in \mathcal{O}_R$.

3. Let $L, N$ be submodules of an $R$-module $M$. Let $E$ be an exact sequence $0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$. If $D$ is a multiplicatively closed subset of $R$, let $D^{-1}E$ be the exact sequence $0 \to D^{-1}L \xrightarrow{D^{-1}\psi} D^{-1}M \xrightarrow{D^{-1}\phi} D^{-1}N \to 0$.

   (i) If $E$ splits, prove that $D^{-1}$ splits.

   (ii) Using (i), quickly show that $D^{-1}(L \oplus N) \simeq D^{-1}L \oplus D^{-1}N$.

   (iii) Quickly show that $D^{-1}(M/L) \simeq D^{-1}M/D^{-1}L$.

4. Let $R = \mathbb{Z}$ and let $M$ be a finite abelian group. Let $D = \mathbb{Z} - (p)$.

   (i) Suppose $q \neq p$ is prime and $M_1 = \mathbb{Z}/q^a$. Show that $D^{-1}M_1 = 0$.

   (ii) If $M_2 = \mathbb{Z}/p^b$, show that $D^{-1}M_2 = M_2$.

   (iii) If $M$ is a finite abelian group, show that $D^{-1}M$ is the Sylow $p$-subgroup of $M$.

5. Let $R = k[x]$.

   (i) Compute $k[x]/(x^2) \otimes_R k[x]/(x^3)$ explicitly.

   (ii) What is $k[x]/(x^n) \otimes_R k[x]/(x^m)$?

# 8 Week 8 - 3/10: Inverse/direct limits

1. Let $V_1 = V(y_1^n - x)$ and $V_2 = V(y_2^m - x)$. For $i = 1, 2$, there are morphisms $f_i : V_i \to \mathbb{P}^1_k$ where $f_i((x, y_i)) = x$. Find an equation for the curve $V_1 \times_{\mathbb{P}^1_k} V_2$. How many irreducible components does it have?

2. Prove that the localization $\mathbb{Z}_{(p)}$ is a direct limit of the form $\overset{\lim}{\to} \mathbb{Z}[1/n]$. Part of the problem is to figure out what the indexing set, partial order, and compatibility morphisms are.

3. Use the universal property to prove that $\mathbb{Z}_p$ is a quotient of $\hat{\mathbb{Z}}$.

4. Prove that $\mathrm{Gal}(\overline{\mathbb{F}}_p, \mathbb{F}_p) \simeq \hat{\mathbb{Z}}$. Find a subfield $L$ of $\overline{\mathbb{F}}_p$ so that $\mathrm{Gal}(L, \mathbb{F}_p) \simeq \mathbb{Z}_p$.

5. Write $2/3$ and $-2/3$ as 5-adic numbers.

6. Let $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$ whose elements are of the form $\alpha = \sum_{i=-m}^{\infty} a_i p^i$. Prove that $\alpha$ is a rational number if and only if the sequence of digits $\{a_i\}$ is periodic (possibly with a finite string before the first period).

# 9 Week 9 - 3/24:

Finish writing up the short project by 4/4 or 4/7.

# 10 Week 10 - 3/31: Dimension theory

1. (i) Let $P$ be a prime ideal of a ring $R$ having unique factorization. Prove $\mathrm{codim}(P) = 1$ if and only if $P = (\pi)$ is a principal ideal generated by an irreducible $\pi \in R$.

   (ii) Let $W \in \mathbb{A}^n_k$ be an irreducible variety. Prove that $W$ is a hypersurface if and only if $\mathrm{codim}(W) = 1$.

2. Let $P$ be a prime ideal of an affine $k$-algebra $A$. Prove $\dim(P) + \mathrm{codim}(P) = \dim(A)$.

3. (i) Let $A$ and $B$ be affine $k$-algebras. Prove that $\dim(A \otimes_k B) = \dim(A) + \dim(B)$.

   (ii) Let $V$ and $W$ be irreducible varieties. Prove $\dim(V \times W) = \dim(V) + \dim(W)$.

4. Consider the ideals $M_1 = (xy - 1)$ and $M_2 = (x, y)$ of the ring $R = k[[x]][y]$.

   (i) Describe the elements of $R$.

   (ii) Show that $M_1$ and $M_2$ are maximal ideals.

   (iii) Show that $\mathrm{codim}(M_1) = 1$ and $\mathrm{codim}(M_2) = 2$.

5. Let $R$ be the non-commutative ring $M_2(k)$ of two-by-two matrices with entries in $k$. Show that $R$ has a composition series of left $R$-modules. (It also has a composition series of right $R$-modules. These two facts imply that $R$ is Artinian.)

6. Let $R$ be the ring of two-by-two matrices

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

   where $a \in \mathbb{Q}$ and $b, d \in \mathbb{R}$. Show that $R$ is not left Artinian.

## 11 Week 12 - 4/14: Final project

Hand in a title, outline, and two references for your final project.

## 12 Week 13 - 4/21: Complete Intersection/Flat

1. (i) Find 6 points $P_1, \ldots, P_6$ in $\mathbb{P}^2$ that are not on a conic.

   (ii) If $V = \{P_1, \ldots, P_6\}$, find $I(V)$.

   (iii) Show that $V$ is not a complete intersection.

   (iv) Show that $V$ is a set-theoretic complete intersection.

2. Let $V = \{(t, t^2, t^3)\} \subset \mathbb{A}^3$ be the twisted cubic.

   (i) Find $\phi_1, \phi_2, \phi_3$ that generate $I(V)$.

   (ii) Show that $V((\phi_1, \phi_2))$ is the union of $V$ with a line $L$.

3. Show that $\mathbb{Q}/\mathbb{Z}$ is not a flat $\mathbb{Z}$-module.

4. This problem shows that the base change of flat modules is flat. Suppose $A$ is a flat $R$-module and $f : R \to S$ where $f(1_R) = 1_S$. Show that $A \otimes_R S$ is a flat $S$-module.

5. I may add another problem about flatness.

Week 11 - 4/7: Differentials, singularities

Week 13 - 4/21: Flatness

Week 14 - 4/28: Deformations

Week 15 - 5/5: Projects

**Grading policy:** 40% project, 40% homework, 10% scribe assignment, 10% short presentation.

**Books:**

Kunz

Dummit/Foote: 15,16

Eisenbud

Matsumura