

Pries: 619 Complex Variables II. Homework 2.

1. Find the zeros and poles of $f(z) = (z^2+3)/(z^4+3z^2+2)$ and check $\sum_{P \in \mathbb{C}_\infty} \text{ord}_P(f) = 0$.
2. Find the degree d of the function $f(z) = z^3/(z^4+27)$. For which c are there less than d values of z so that $f(z) = c$? Describe the nature of the branching above these points.
3. If $f(z) = p(z)/q(z)$ is a rational function on \mathbb{C} , show that $\text{ord}_\infty(f) = \deg(q) - \deg(p)$. Show that $\sum_{P \in \mathbb{C}_\infty} \text{ord}_P(f) = 0$.
4. Give an example of a meromorphic function on \mathbb{C} with a simple pole at each $z \in \mathbb{Z}$. Does it extend to a meromorphic function on \mathbb{C}_∞ ?
5. Find formulae for the 6 automorphisms of \mathbb{C}_∞ which fix the set $\{0, 1, \infty\}$.
6. Suppose $f(z) = (az + b)/(cz + d)$ is an automorphism of \mathbb{C}_∞ with $a, b, c, d \in \mathbb{R}$. Find a condition on a, b, c, d which is equivalent to the condition that f stabilizes the upper half plane.
7. Let $f(z)$ be a rational function so that $|z| = 1$ implies $|f(z)| = 1$. Show that $f(\alpha) = 0$ if and only if $f(1/\bar{\alpha}) = \infty$ and thus find the most general form of $f(z)$.