Pries: 619 Complex Variables II. Homework 12 (Last one!)

1. Show that $\mathrm{PSL}_2(\mathbb{R})$ acts doubly transitively on $\mathbb{R}_\infty$.

2. In this problem, we find a formula for the hyperbolic distance between two points P and Q in the upper half plane. Let M and N be the two endpoints of the hyperbolic line containing P and Q (so that P is between M and Q). Let f be an isometry so that $f(P) = i$, $f(M) = 0$, and $f(N) = \infty$. Let $f(Q) = x + iy$.

   i) What is x?

   ii) Compare y to the hyperbolic distance d between P and Q.

   iii) Use the cross-ratio to find another formula for y.

   iv) Prove $d = |\ln(\mathrm{CR}(Q, P, M, N)|$.

   v) Find the hyperbolic distance between P= -4+6i and Q=4+6i.

3. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $r = \gamma(\infty) \in \mathbb{Q}$. Prove that γ sends the open neighborhood $\{\tau \in \mathbb{H} \mid \mathrm{im}(\tau) > C\}$ to the interior of a circle in $\mathbb{H}$ tangent to the real axis at r.

4. Let D be a small neighborhood of $i \in \mathbb{H}$. Let D' be a small disk in $\mathbb{C}$ centered at 0. Consider the map $f : D \to D'$ sending $z \mapsto (z-i)/(z+i)$. Let S be the transformation $S(z) = -1/z$ on D. Let $\overline{S}$ be the transformation $\overline{z} = -z$ on D'

   i) Show that $f \circ S = \overline{S} \circ f$.

   Background: Recall that this implies that f descends to a map $\overline{f} : D/S \to D'/\overline{S}$. Also $z \to z^2$ gives a map $\iota : D'/\overline{S} \to D''$ where $D'' \in \mathbb{C}$ is another small disk. The map $\iota \circ \overline{f}$ takes the orbit of $\{z, S(z)\}$ to $((z-i)/(z+i))^2$.

   ii) What is the largest disk D centered at $i \in \mathbb{H}$ for which $\iota \circ \overline{f}$ is injective?

5. Show that the genus of $X(2)$ is 0 using the method of triangulation.

6. Consider $\Gamma = \Gamma_1(3)$.

   i) What is the index $c = [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma]$?

   i) Find representatives γ_i of the cosets of Γ so that $D_\Gamma = \cup_{i=1}^{c} \gamma_i D$ is connected.

   ii) Draw D_Γ.

   iii) Explicitly describe the points where the map $X_1(3) \to \mathbb{P}^1$ is not c-to-1.

   iv) Find the genus of $X_1(3)$.

7. Show that $\Gamma(N)\backslash\mathbb{H}$ is a moduli space parametrizing elliptic curves along with a choice of basis of the N-torsion $E[N]$.