Pries: 619 Complex Variables II. Homework 2. Due Wednesday 9/11

Discussion session 2:

What is an elliptic curve by Daniels and Lozano-Robledo.
https://www.ams.org/journals/notices/201703/rnoti-p241.pdf

Homework problems:

1. If $f(z) = p(z)/q(z)$ is a rational function on $\mathbb{C}$, show that $\mathrm{ord}_\infty(f) = \deg(q) - \deg(p)$. Show that $\sum_{P \in \mathbb{C}_\infty} \mathrm{ord}_P(f) = 0$.

2. Given points $z_1, \ldots, z_r \in \mathbb{C}_\infty$ and integers $n_1, \ldots, n_r$ such that $\sum_{i=1}^r n_i = 0$, prove there exists a meromorphic function $f(z)$ such that $\mathrm{div}(f) = \sum_{i=1}^r n_i \cdot z_i$.

3. Give an example of a meromorphic function on $\mathbb{C}$ with a simple pole at each $z \in \mathbb{Z}$. Does it extend to a meromorphic function on $\mathbb{C}_\infty$?

4. * Let $f(z)$ be a rational function so that $|z| = 1$ implies $|f(z)| = 1$. Show that $f(\alpha) = 0$ if and only if $f(1/\overline{\alpha}) = \infty$ and thus find the most general form of $f(z)$.

5. Let $f(x, y) = x^3 + y^3 + 1 - txy$. Find the values of t for which V_f is not smooth.

6. Let $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^n - x\}$. The implicit function theorem implies that, near $(0, 0)$, X is the graph of $y = g(x)$ for some function $g(x)$ which is holomorphic near 0. Find the first several terms of the Taylor series of $g(x)$.

7. Let $f(z) = 4P(z)^3 - 60G_4P(z) - 140G_6$, where G_4, G_6 are defined from L, as in the lecture.

   i) Show that $P'(z)^2 - f(z)$ is holomorphic near $z = 0$ and has a zero at $z = 0$ by looking at its Laurent series.

   ii) Explain why $P'(z)^2 - f(z)$ is holomorphic everywhere.

   iii) Explain why $P'(z)^2 - f(z)$ is the zero-function.

8. As in the previous problem, let $f(z) = 4P(z)^3 - 60G_4P(z) - 140G_6$. Let $E \subset \mathbb{P}^2$ be the projective curve given by the affine equation $y^2 = f(z)$. Show that E is non-singular and that there is an isomorphism $\tau : \mathbb{C}/L \to E$.