Pries: 619 Complex Variables II. Homework 1. Due Wednesday 9/4

Discussion session 1:

Why ellipses are not elliptic curves: Adrian Rice/Ezra Brown, Mathematics Magazine volume 85, pages 163-176, 2012

Homework problems:

1. Miranda I.1.F. Let U be an open subset of $X = \mathbb{R}^2$. Show that these two coordinate charts are not compatible: $\phi_1(x, y) = x + iy$ and
$$\phi_2(x, y) = \frac{x}{1 + \sqrt{x^2 + y^2}} + i\frac{y}{1 + \sqrt{x^2 + y^2}}.$$
2. Miranda I.1.G. Show that the transition function T for the two stereographic projection maps is holomorphic and that $T(z) = 1/z$ (hint - avoid working with inverse maps).
3. Show that the reflection ρ_y in S^2 over the plane $y = 0$ corresponds (via stereographic projection ϕ_0) to complex conjugation cc in $\mathbb{C}$: $cc \circ \phi_0 = \phi_0 \circ \rho_y$.
4. The eight points with coordinates $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ form the vertices of a cube inside the sphere. Find the images of these points under stereographic projection. Explain why the automorphism group of $\mathbb{C}_\infty$ contains a subgroup isomorphic to S_4.
5. Automorphisms of $\mathbb{C}_\infty$ are of the form $f(z) = (az + b)/(cz + d)$ where $a, b, c, d \in \mathbb{R}$. Prove that $f(z)$ is invertible iff $ad - bc \neq 0$ and that $f(z)$ stabilizes the upper half plane iff $ad - bc > 0$.
6. Miranda I.2 F. Show that the group law on a complex torus X is *divisible*: given $P \in X$ and an integer $n \geq 1$, there is a point $Q \in X$ such that $nQ = P$. In fact there are exactly n^2 such points.

For problem 2, recall that $S^2 : x^2 + y^2 + z^2 = 1$, $U_0 = S^2 - NP$ and $U_\infty = S^2 - SP$. Also $T = \phi_0 \circ \phi_\infty^{-1}$, where

$$\phi_0 : U_0 \to \mathbb{C},\ \phi_0((x, y, z)) = \frac{x}{1 - z} + i\frac{y}{1 - z}.$$

$$\phi_\infty : U_\infty \to \mathbb{C},\ \phi_\infty((x, y, z)) = \frac{x}{1 + z} + i\frac{-y}{1 + z}.$$