

**Pries: 619 Complex Variables II. Homework 3.** Due Tuesday 10/1

**Computational**

1. Find formulae for the 6 automorphisms of  $\mathbb{C}_\infty$  which fix the set  $\{0, 1, \infty\}$ .
2. Find the degree  $d$ , the critical points, and the multiplicities of the critical points for the function  $f(z) = z^3/(z^4 + 27)$ .
3. Miranda II.3 G. Prove that composition of automorphisms of  $\mathbb{C}_\infty$  is the same as matrix multiplication in  $\text{GL}_2(\mathbb{C})$ .
4. Give an example of a meromorphic function on  $\mathbb{C}$  with a simple pole at each  $z \in \mathbb{Z}$ . Does it extend to a meromorphic function on  $\mathbb{C}_\infty$ ?

**Thoughtful**

1. Given points  $z_1, \dots, z_r \in \mathbb{C}_\infty$  and integers  $n_1, \dots, n_r$  such that  $\sum_{i=1}^r n_i = 0$ , prove there exists a meromorphic function  $f(z)$  such that  $\text{div}(f) = \sum_{i=1}^r n_i \cdot z_i$ .
2. Consider the automorphisms  $\sigma_n(z) = \zeta_n z$  and  $\tau(z) = \frac{1}{z}$  of  $\mathbb{C}_\infty$ . Prove that the group generated by  $\sigma_n$  and  $\tau$  is isomorphic to  $D_n$ . Find a function  $w = g(z)$  which is invariant under  $\sigma_n$  and  $\tau$  such that the degree of the field extension  $\mathbb{C}(z)/\mathbb{C}(w)$  is  $2n$ . Find all the ramification points and their ramification indices.
3. Miranda II.3.K Let  $L$  be a lattice in  $\mathbb{C}$  and let  $0 \neq \alpha \in \mathbb{C}$ . Prove that the map  $\phi : \mathbb{C}/L \rightarrow \mathbb{C}/(\alpha L)$  sending the coset  $z + L$  to the coset  $\alpha z + \alpha L$  is a well-defined biholomorphic map. Show that every torus  $\mathbb{C}/L$  is isomorphic to a torus of the form  $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$  for some  $\tau \in \mathbb{C}$  such that  $\text{Im}(\tau) > 0$ .
4. \* Let  $f(z)$  be a rational function so that  $|z| = 1$  implies  $|f(z)| = 1$ . Show that  $f(\alpha) = 0$  if and only if  $f(1/\bar{\alpha}) = \infty$  and thus find the most general form of  $f(z)$ .