Pries: 619 Complex Variables II. Homework 10.

1. Let X be the elliptic curve with equation $y^2 = x^3 - 1$. Let $P_0 = (1, 0)$. Using a computer, find the lattice L so that $X \simeq \mathbb{C}/L$. Hint: using ideas from class and Miranda VIII.1.C, parametrize lines between the branch points and integrate $dx/\sqrt{x^3 - 1}$ over these lines. My first attempt at this crashed the computer. If this happens, try working with the equation $y^2 = (x + 1)^3 - 1$ and the point $(0, 0)$.

2. Let X be the elliptic curve given by the equation $y^2 = x^3 - 1$. Without using a computer, show that the lattice L in problem 1 must be hexagonal. In other words, show that the lattice is invariant under multiplication by $e^{2\pi i/3}$.

3. Miranda VIII.1 D

4. What are the automorphisms of the Riemann surface with equation $y^2 = x^6 - 1$? How are these automorphisms visible in terms of the lattice found in problem 3?

5. Let X be a Riemann surface of genus 2. Let $P_0 \in X$. Let $X^{(2)}$ be the set of unordered pairs $\{P_1, P_2\}$ of points of X. Let $\sim$ denote linear equivalence of divisors on X.

   i) Given $P_1, P_2, Q_1, Q_2 \in X$, show there exist $R_1, R_2 \in X$ so that $P_1 + P_2 + Q_1 + Q_2 \sim R_1 + R_2 + 2P_0$.

   ii) Show that the pair $\{R_1, R_2\}$ in (i) is unique if and only if $P_1+P_2+Q_1+Q_2 \not\sim 2P_0+K$ (where K is the canonical divisor).

   iii) Define an equivalence relation on $X^{(2)}$:
   $\{P_1, P_2\} \equiv \{Q_1, Q_2\}$ iff either $\{P_1, P_2\} = \{Q_1, Q_2\}$ or $P_1 + P_2 \sim Q_1 + Q_2 \sim K$.
   Let $J = X^{(2)}/\equiv$ and let $[P_1, P_2]$ denote the image of $\{P_1, P_2\}$ in J.
   Define a binary operation $\oplus$ on J:
   $[P_1, P_2] \oplus [Q_1, Q_2] = [R_1, R_2]$ iff $(P_1 + P_2 - 2P_0) + (Q_1 + Q_2 - 2P_0) \sim R_1 + R_2 - 2P_0$.
   Show that $\oplus$ is well-defined and that it gives J the structure of an abelian group.

   iv) Show that $J \simeq \mathrm{Pic}^0(X)$.

6. Narrow down a project topic.