Pries 605A Algebraic Number Theory spring 2016.
Homework 6.

Due Fri 4/1: Choose a project topic. I'm happy to help brainstorm topics.
Due Fri 4/8: Let d_1, d_2 be distinct square-free integers.

1. Under what conditions on d_1 does the prime 2 ramify, split, or stay inert in $\mathbb{Q}(\sqrt{d_1})$?

2. Let L be the bi-quadratic field $L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$.

   (i) Find an example of d_1, d_2, p such that (p) is totally ramified in $\mathcal{O}_L$.

   (ii) Find an example of d_1, d_2, p such that (p) is totally split in $\mathcal{O}_L$.

   (iii) Prove that it is not possible for (p) to be totally inert in $\mathcal{O}_L$.

   (iv) Show that no primes of $\mathbb{Z}$ are totally inert in $\mathbb{Q}(\zeta_8)$.

3. If $L|K$ is a Galois extension of number fields whose Galois group is non-cyclic, show there are at most finitely many prime ideals for K which are non-split in L.

4. There is a correspondence between primes of a ring and valuations. For example, the prime p of $\mathbb{Z}$ corresponds to the p-adic absolute value. There is one other valuation on $\mathbb{Z}$, namely the absolute value function $|\cdot|$. This means that $\mathbb{Z}$ has one other "prime", the prime at infinity, which we denote by p_∞. The analogue of the residue field $\mathbb{Z}/p$ is the completion of $\mathbb{Z}$ for $|\cdot|$, namely $\mathbb{R}$.

   A. Under what conditions on d does p_∞ ramify, split, or stay inert in $\mathbb{Q}(\sqrt{d})$?

   B. Describe the factorization of p_∞ in $\mathbb{Z}[2^{1/3}]$.

5. Let $K = \mathbb{C}(x)$ and $L = K[y]/(y^2 - f(x))$. Describe the primes of $\mathbb{C}[x]$ which are ramified in L when $f(x) \in \mathbb{C}[x]$ is a polynomial with no multiple roots.

6. Let $K = \mathbb{C}(x)$ and $L = K[y, z]/(y^2 - \pi x, z^3 - (y + \sqrt{\pi})/(y - \sqrt{\pi}))$

   A. Show that $\mathrm{Gal}(L : K) = S_3$.

   B. Describe the primes of $\mathbb{C}[x]$ which are ramified in L.