Pries: 566 Abstract Algebra,
Final: Tuesday, December 13, 2011

Name:________________________

This midterm has 18 short problems each worth 4 points and 2 long answer problems each worth 18 points. You may skip one short problem on groups and one short problem on rings. Show all your work and explain all your answers to receive full credit.

For each short problem, either state that it exists (E) or that it does not exist (DNE). If it exists, give an example and a brief explanation of why it is an example. If it does not exist, give a brief explanation/proof of the reason.

Short Problems on Groups:

1. An element of order 6 in A_7.

2. An element of infinite order in $\mathrm{GL}_2(\mathbb{R})$.

3. An isomorphism $\phi : \mathbb{Z} \to \mathbb{Q}$.

4. A conjugacy class of size 6 in S_4.

5. A faithful group action of D_{16} on the set $\{1, 2, 3, 4, 5\}$.

6. A non-normal subgroup of $Q_8 \times \mathbb{Z}/4$.

7. Two non-isomorphic abelian groups of order 70.

8. Two non-isomorphic groups of order 39.

9. A group of size 60 with four Sylow 5-subgroups.

Short Problems on Rings:

1. A unit $u \neq \pm 1$ in $\mathbb{Z}[\sqrt{3}]$.

2. A zero divisor in $M_2(\mathbb{R})$.

3. An ideal I of $R = (\mathbb{Z}/7)[x]$ such that $\#(R/I) = 28$.

4. A monic degree 2 polynomial in $(\mathbb{Z}/8)[x]$ with more than 2 roots.

5. A finite set of generators for the ideal $I = \{f(x, y) \in \mathbb{C}[x, y] \mid f(3, 4) = 0\}$.

6. An integer n such that $\langle n \rangle = \langle 2, 1+\sqrt{-5} \rangle \langle 2, 1-\sqrt{-5} \rangle$ as ideals of $\mathbb{Z}[\sqrt{-5}]$.

7. An inverse of $1 \cdot (1,2,3)$ in the group ring $\mathbb{Z}[S_3]$.

8. A unique factorization domain which is not a principal ideal domain.

9. A non-zero prime ideal of the ring $\mathbb{Q}[x]$ which is not maximal.

Long Answer Problems:
Strive for clarity. You can use earlier parts of a problem even if you haven't solved them.

1. A non-abelian group of order p^3: Let $H = C_{p^2}$ be the (multiplicative) cyclic group of order p^2 with generator r. For $1 \leq i \leq p^2$, let $\tau_i : C_{p^2} \to C_{p^2}$ be the homomorphism of H given by $\tau_i(r) = r^i$. Recall that τ_i is an isomorphism if and only if $\gcd(i, p^2) = 1$.

   a. Prove that $\mathrm{Aut}(H)$ contains an automorphism of order p.

   b. Let $K = C_p$ be the cyclic group of order p with generator k. Prove that there exists a non-trivial homomorphism $\phi : K \to \mathrm{Aut}(H)$.

   c. Prove that the semi-direct product $G = H \rtimes_\phi K$ is a non-abelian group of order p^3, containing an element of order p^2.

   d. Find a value of i such that τ_i is an automorphism of order p and use it to find a presentation for G.

2. Quotient rings: Suppose R is a commutative ring and $I \subset R$ is an ideal.

   a. Prove that R/I is an integral domain if and only if I is a prime ideal.

   b. Show that $\langle 13 \rangle$ is not a prime ideal of $\mathbb{Z}[i]$.

   c. Prove that $\langle 4 + i \rangle$ is a prime ideal of $\mathbb{Z}[i]$.

   d. Briefly explain why $\mathbb{Z}/\langle 4 + i \rangle$ is not just an integral domain, but also a field.