Pries: M405 - Number Theory, spring 2020
Homework 2: due 3/13

Do 10 parts out of the 6 parts of problem 1 and the 8 parts of problem 2. For extra credit, do some of the remaining 4 parts.

1. Let $R = \mathbb{Z}[\sqrt{-6}]$. Let $P = \langle 2, \sqrt{-6} \rangle$ and let $Q = \langle 3, \sqrt{-6} \rangle$.

   (a) Show that 2, 3, and $\sqrt{-6}$ are irreducible in R.

   (b) Prove that P and Q are non-principal ideals by showing that neither has a generator.

   (c) Show that $P^2 = \langle 2 \rangle$, $Q^2 = \langle 3 \rangle$, and $PQ = \langle \sqrt{-6} \rangle$.

   (d) Show that the element 6 does not factor uniquely into irreducible elements of R but that the ideal $\langle 6 \rangle$ has a unique factorization into ideals of R.

   (e) Find an element $\gamma \in \mathbb{Q}(\sqrt{-6})$ such that $P = \gamma Q$.

   (f) Show that the class group of R has size two. Hint: If W is a non-principal ideal of R, let α be an element of smallest non-zero norm in W. Show that $W = \langle \alpha, \alpha\sqrt{-6}/2 \rangle$ and that $W \sim P$.

2. Let g be an odd positive integer. Suppose there is an odd integer x, such that $x^2 < 3^g/2$ and $d = 3^g - x^2$ is square-free. Redo problems 1-8 from the Week 7 Friday class handout (attached below) in this more general case to show that the class group of $\mathbb{Z}[\sqrt{-d}]$ has an element of order g.

   For the last step: to show $m = g$, write $P^m = \langle u + v\sqrt{-d} \rangle$; find a formula for 3^m; break into cases for $v = 0$ and $v \neq 0$ and get a contradiction if $m < g$ in each case.

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Handout 7F: A non-principal ideal P such that P^3 is principal

Let $g = 3$. Let $x = 1$. Find $d = 3^g - x^2$. Let $R = \mathbb{Z}[\sqrt{-d}]$.

1. Show that 3 is irreducible in R.
2. Let $z = x + \sqrt{-d}$. Show that $3^g = z\bar{z}$.
3. Show that z and $\bar{z}$ are relatively prime;
(there is no $t \in R$ which divides both of them other than $t = \pm 1$).
4. Explain why the g factors of 3 can't be split up between z and $\bar{z}$.
5. Let $P = \langle 3, x + \sqrt{-d} \rangle$. Show that P is a non-principal ideal;
(there is no $\tau \in R$ such that $P = \langle \tau \rangle$).
6. Show that $\langle 3 \rangle = P\bar{P}$.
7. Show that $P^g = \langle x + \sqrt{-d} \rangle$ using parts 2 and 6. This means that P^g is principal.
8. Let m be the smallest positive number such that P^m is principal. Show that $m = g$.
Step 1: m divides g.