Pries: M405 - Number Theory, spring 2020
Homework 1: due 2/7

1. In Section 1.7 Stillwell, it shows how to find a formula for all rational points on the circle $x^2 + y^2 = 1$ using the Diophantus chord method. In this problem, the goal is to find a formula for all rational points on the hyperbola H with equation $x^2 - y^2 = 1$ using the same method.

   (a) Find the equation for the line L with slope t going through the point $(-1, 0)$.

   (b) Find the quadratic equation for x (similar to top of page 15) whose roots are -1 and the 2nd point of intersection of L and H.

   (c) Find the x and y coordinate of this point of intersection.

   (d) Draw a graph of this procedure.

   (e) What is different about the cases $|t| < 1$, $|t| > 1$, and $t = \pm 1$?

   (f) Explain why every rational point on H is found with this method.

2. Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the Gaussian integers. If $z = a + bi$ is in $\mathbb{Z}[i]$, let $\bar{z} = a - bi$ be its complex conjugate. The norm of z is $N(z) = a^2 + b^2$ and the trace of z is $Tr(z) = 2a$.

   (a) Write $w = a' + b'i$. Show that $N(zw) = N(z)N(w)$ and $Tr(z+w) = Tr(z)+Tr(w)$.

   (b) Show that z is a root of the equation $X^2 - Tr(z)X + N(z) = 0$.

   (c) Find all possible $z \in \mathbb{Z}[i]$ such that $N(z) = 1$.

   (d) * If p is a prime such that $p \equiv 3 \bmod 4$, show that $p \neq N(z)$ for any $z \in \mathbb{Z}[i]$.

   (e) * Suppose $Z = \alpha + \beta i$ where α, β are fractions. Use the same formulas for N and Tr. If $N(Z)$ and $Tr(Z)$ are both integers, show that α and β are integers.