Power series & Taylor series: Review exercises

~~1~~. Let $f(x) = \begin{cases} e^x + e^{-1/x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ and assume that $f^{(n)}(0) = 1$ for $n = 0, 1, 2, 3, \ldots$.

(a) What is the Maclaurin series for f?

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

(b) What is the interval of convergence of the Maclaurin series?

ratio test $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim \frac{|x|}{n+1} = 0 < 1$

converges everywhere

(c) For what values of x does $f(x)$ equal the sum of the Maclaurin series?

nowhere

(this problem is a little odd)

$T(x)$ converges to ~~...~~ e^x.

Ⓐ identifying Taylor series
Ⓑ finding Taylor series
Ⓓ error
Ⓒ manipulating Taylor series
Ⓔ interval of convergence

$\boxed{2b}$	geometric series + interval	Ⓔ
$\boxed{3a}$	binomial series	Ⓒ Ⓐ <u>more</u>
$\boxed{3b}$	integrating	Ⓒ
$\boxed{4}$	error	Ⓓ
$\boxed{5a}$	finding Taylor for $\ln(x)$	Ⓑ
$\boxed{6}$	interval + alt series test	Ⓔ

1 weird, 2 messy, 5b type, 7

2. (a) Find the Maclaurin series for the function $f(x) = x\cos^2 x$. Write the result in closed form—as a sum $\sum_{n=0}^{\infty} a_n x^n$. (Hint: $\cos^2 x = \frac{1}{2}(1+\cos(2x))$.)

$$\cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}$$

$$x\left(\frac{1+\cos(2x)}{2}\right) = \left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{2^{2n}}{2} x^{2n}\right)x = x + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n-1} x^{2n+1}$$

$$f(x) = \cancel{\left(\sum_{n=} \frac{(-1)^{n-1} 2^{2n-2}}{(2n-2)!} x^n\right)} \quad f(x) = \sum_{m=0}^{\infty} a_m x^m \quad \text{where } a_m = \begin{cases} 0 & m \text{ even} \\ \frac{3}{2} & m=1 \\ \frac{(-1)^{\frac{m-1}{2}} 2^{m-2}}{(m-1)!} & m \text{ odd} \end{cases}$$

$m = 2n+1$

(b) Find the sum of the series $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ as a function of x. What is the interval of convergence of the series.

geometric series

ratio $r = \frac{x^2+1}{3}$ first term 1

sum $\frac{a}{1-r} = \cancel{\frac{3}{x^2+1}}$ $\frac{1}{1-\left(\frac{x^2+1}{3}\right)} = \boxed{\frac{3}{2-x^2}}$

converges if $\left|\frac{x^2+1}{3}\right| < 1 \iff |x^2+1| < 3$

$\iff -3 < x^2+1 < 3 \iff -4 < x^2 < 2$

$\iff -\sqrt{2} < x < \sqrt{2}$

endpoints $x = \sqrt{2}$ $\sum 1$ diverges

$x = -\sqrt{2}$ $\sum 1$ diverges

interval $(-\sqrt{2}, \sqrt{2})$

$\boxed{3}$ (a) Find the first 5 terms of the Maclaurin series of the function $f(x) = (1-3x^2)^{-1/3}$.

binom $(1+t)^k = 1 + kt + \frac{k(k-1)}{2}t^2 + \frac{k(k-1)(k-2)}{3!}t^3 + \frac{k(k-1)(k-2)(k-3)}{4!}t^4$

$(1+t)^{-1/3} = 1 - \frac{1}{3}t + \frac{(-1/3)(-4/3)}{2}t^2 + \frac{(-1/3)(-4/3)(-7/3)t^3}{3!} + \frac{(-1/3)(-4/3)(-7/3)(-10/3)t^4}{4!}$

$(1-3x^2)^{-1/3} = 1 + x^2 + \frac{1\cdot 4}{2}x^4 + \left(\frac{1\cdot 4\cdot 7}{3!}\right)x^6 + \frac{1\cdot 4\cdot 7\cdot 10}{4!}x^8$

$+ \cdots$

(b) If the first 6 terms of the Maclaurin series of the function $f(x) = \frac{1}{\sqrt{1-x^2}}$ are

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \frac{63}{256}x^{10} + \cdots,$$

find the first 7 terms of the Maclaurin series of the function $f(x) = \arcsin(x)$.

$\arcsin(x) = \int \frac{1}{\sqrt{1-x^2}}\,dx = x + \frac{1}{3!}x^3 + \frac{3}{5\cdot 8}x^5 + \frac{5}{16\cdot 7}x^7 + \frac{35}{128\cdot 9}x^9 + \frac{63}{256\cdot 11}x^{11}$

first coeff 0

~~5~~ 4. The Maclaurin series for $f(x) = \cos x$, $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, converges for all x, $-\infty < x < \infty$. Show that the series converges to $f(x) = \cos x$ for all x.

$$|R_n(x)| \le \frac{M}{(n+1)!}|x|^{n+1}$$

$$M = \max\{\pm\cos x, \pm\sin x \mid \text{any } x\} = 1$$

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} |R_n(x)| = 0$$

5 (a) Find the Taylor series expansion for $f(x) = \ln(x)$ at $a = 4$. Write the result using summation notation.

n	$f^n(x)$	$f^n(4)$
0	$\ln x$	$\ln(4)$
	$\frac{1}{x}$	$\frac{1}{4}$
	$-\frac{1}{x^2}$	$-\frac{1}{4^2}$
	$\frac{2}{x^3}$	$\frac{2}{4^3}$
4	$-\frac{6}{x^4}$	$-\frac{6}{4^4}$
n		$\frac{(-1)^{n-1}(n-1)!}{4^n}$

$$\boxed{\ln(4) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!\,4^n}(x-4)^n}$$

$$\boxed{T(x) = \ln(4) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,4^n}(x-4)^n}$$

(b) Find the first four terms of the Taylor series expansion for $f(x) = x^2 \ln(x)$ at $a = 4$.

$$\cancel{(\ln 4)x^2 + \frac{1}{4}x^2}$$

$$\left((x-4)^2 + + 8(x-4) + 16\right)\ln x$$

$x^2 - 8x + 16 \quad 8x - 32$

what a mess!

typo?

Correct answer involves multiplying $(x-4)^2 + 8(x-4) + 16$ by answer to 5a

5b

this problem probably has a typo.
The answer is not

$x^2 \cdot$ (answer for 5a)

because this wouldn't be a power series centered at $a = 4$

6. For what values of x does the power series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ converge.

ratio test $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim \frac{|x|^{2n+1}}{2n+1} \frac{2n-1}{|x|^{2n-1}} = |x|^2$

$\rho < 1$ when $-1 < x < 1$

$x = 1$ $\quad \sum (-1)^{n-1} \frac{1}{(2n-1)}$ converges by AST

$x = -1$ $\quad \sum \frac{(-1)^{3n-2}}{2n-1}$ converges by AST $\quad -1 \le x \le 1$

$[-1, 1]$

7. Find the Taylor series of $f(x) = x^3 - 2x + 4$ at $a = 2$.

$(x-2)^3 = x^3 - 3 \cdot 2x^2 + 3x2^2 - 2^3$ $\quad 12 \quad -8$

$+3 \cdot 2(x-2)^2 = 3\,2\,x^2 - 3 \cdot 2^3 x + 6 \cdot 2^2$ $\quad -24 \quad +24$

$10(x-2)$ $\quad 10x - 20$ $\quad -20$

unusual method.

$\cancel{+16} \cancel{+20} + 8$

$$\boxed{\cancel{20}\, 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3}$$

standard method: Find derivatives at $a = \cancel{0}\,2$

$x^3 - 2x + 4$	8
$3x^2 - 2$	10
$6x$	12
6	6

$$\boxed{8 + 10(x-2) + \frac{12}{2!}(x-2)^2 + \frac{6}{3!}(x-2)^3}$$