Review Exercises: Series

1. Determine whether the series is convergent or divergent. In either case explain why. **You must justify your answers.**

(a) $\sum_{n=1}^{\infty} \frac{2^n}{(2n+1)!}$

ratio test $\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim \frac{2^{n+1}/(2n+3)!}{2^n/(2n+1)!} = \lim_{n\to\infty} \frac{2}{(2n+3)(2n+2)} = 0$

$\rho < 1 \implies$ series converges

(b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+n+1}}$

$a_n = \frac{n}{\sqrt{n^3+n+1}}$ $\qquad b_n = \frac{1}{n^{1/2}}$

$\sum b_n$ diverges by p-series test $p = 1/2 < 1$

Limit comparison test

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n \cdot n^{1/2}}{\sqrt{n^3+n+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}+\frac{1}{n^3}}} = \frac{1}{\sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n^2}+\frac{1}{n^3}\right)}}$

$= \frac{1}{\sqrt{1}} = 1 \implies$ $\sum a_n$ has same behavior as $\sum b_n$

$\implies \sum a_n$ diverges.

(c) $\sum_{n=1}^{\infty} ne^{-n^2}$

$f(x) = xe^{-x^2}$

integral test $\int_1^{\infty} f(x)\,dx \overset{\lim_{b\to\infty}}{=} \frac{1}{-2} e^{-x^2} \Big|_1^b = \lim_{b\to\infty} \frac{1}{-2} e^{-b^2} + \frac{1}{2} e^{-1}$

$= 0 + \frac{1}{2e}$ finite.

so $\sum_{n=1}^{\infty} ne^{-n^2}$ converges

(d) $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

ratio test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)!/(n+1)3^{n+1}}{n!/n3^n} = \lim_{n\to\infty} (n+1) \frac{1}{3} \frac{n}{n+1} = \infty$

$\rho > 1$ so $\sum \frac{n!}{n3^n}$ diverges

(e) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+n+1}}$

$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} = \frac{1}{\sqrt{\lim_{n\to\infty}(1+\frac{1}{n}+\frac{1}{n^2})}} = \frac{1}{\sqrt{1}} = 1$

$1 \neq 0$

diverges by test for divergence

2. Determine whether the series is absolutely convergent, conditionally convergent or divergent. In any case, **explain why.**

(a) $\sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n+2^n}$

$a_n = \frac{1}{n+2^n} < b_n = \frac{1}{2^n}$ $\sum b_n = \sum \left(\frac{1}{2}\right)^n$ converges geometric series w/ $r = \frac{1}{2}$

so $\sum a_n$ converges by comparison test.

so $\sum \frac{(-1)^{n+1}}{n+2^n}$ absolutely converges.

(b) $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

$\lim_{n\to\infty} 2^{1/n} = 1 \Rightarrow \lim_{n\to\infty} (-1)^n 2^{1/n}$ doesn't exist.

$\Rightarrow \sum (-1)^n 2^{1/n}$ diverges

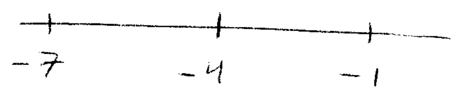
3. Determine the radius of convergence and the interval of convergence (including the endpoints) of the series $\sum_{n=1}^{\infty} \frac{(x+4)^n}{n3^n}$.

root test $\quad \sqrt[n]{|a_n|} = \frac{|x+4|}{\sqrt[n]{n} \cdot 3}$

$$\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \frac{|x+4|}{1 \cdot 3}$$

$$\rho < 1 \iff |x+4| < 3$$

$$\iff -7 < x < -1$$



center $\quad x = -4$
radius $\quad 3$

endpoints $\quad x = -7 \qquad \sum \frac{-3^n}{n(3^n)} = \sum \frac{(-1)^n}{n}$

alternating harmonic series $\qquad$ converges

$x = -1 \qquad \sum \frac{3^n}{n3^n} = \sum \frac{1}{n} \qquad$ harmonic series

diverges

interval $\quad [-7, -1)$

4. (a) Derive a formula (a closed form formula, not one that is the sum of n terms) for the Nth partial sums, s_N, of the series $\sum_{n=3}^{\infty} 2(0.4)^{2+n}$.

$$S_N = 2(.4)^5 + 2(.4)^6 + \ldots 2(.4)^{N+2}$$

$$.4\, S_N = 2(.4)^6 + \ldots 2(.4)^{N+2} + 2(.4)^{N+3}$$

$$S_N(1-.4) = 2(.4)^5 + 0 \ldots + 0 - 2(.4)^{N+3}$$

$$S_N(.6) = 2(.4)^5 - 2(.4)^{N+3}$$

$$\boxed{S_N = 2\left(\tfrac{5}{3}\right)\left(.4^5 - .4^{N+3}\right)}$$

(b) If the sum of this series exists (i.e. if the series converges), then find the sum of the series using the formula for s_N (find an exact answer—no calculator). If the series does not coverge, then use the formula for s_N to explain why.

$$\lim_{N\to\infty} S_N = 2\left(\tfrac{5}{3}\right)\left(.4^5 - \lim_{n\to\infty}(.4)^{N+3}\right)$$

$$S = 2\cdot\tfrac{5}{3}(.4^5)$$

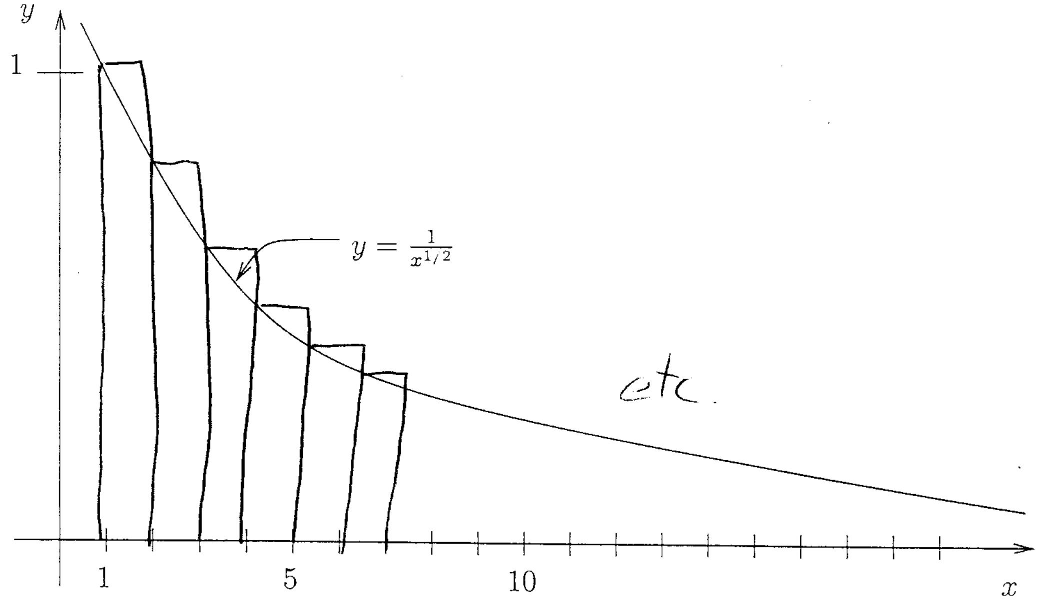
~~some~~ so $\sum_{n=3}^{\infty} 2(.4)^{2+n}$ conveges to

$$S = 2\left(\tfrac{5}{3}\right)(.4^5)$$

5. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ and the function $f(x) = \frac{1}{\sqrt{x}}$, which is graphed below. Illustrate the proof of the integral test by drawing appropriate rectangles on the graph to deduce that

$$\sum_{i=1}^{n} a_i = \frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \cdots + \frac{1}{n^{1/2}} \geq \int_1^{n+1} \frac{1}{x^{1/2}}\,dx.$$

Explain in your own words how this shows that the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is divergent.



sum of series = sum of areas of rectangles

~~Area~~ $\geq$ sum of areas under curves

$$\sum_{n=1}^{N} \frac{1}{n^{1/2}} \geq \int_1^{N+1} f(x)\,dx$$

$$\lim \int_1^{N+1} f(x) = \infty$$

diverges

$$\text{so } \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^{1/2}} \text{ diverges}$$

$$\text{so } \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \text{ diverges.}$$

6. Assume that the function y can be written as a power series

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Use this series to find a solution to the initial value problem (differential equation)

$$y - y' = 3 \text{ with } y(0) = 4.$$

Write your solution for y as a power series and then determine the function that is represented by the series (you should be able to see which function it is by the form of the power series).

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$y - y' = (a_0 - a_1) + (a_1 - 2a_2) + (a_2 - 3a_3) + \cdots$$

$$y(0) = 4 \implies \boxed{a_0 = 4}$$

$$y - y' = 3 \implies \boxed{a_1 = 1}$$

$$a_1 - 2a_2 = 0 \implies \boxed{a_2 = \frac{1}{2}}$$

$$a_2 - 3a_3 = 0 \implies \boxed{a_3 = \frac{1}{2 \cdot 3}}$$

$$a_3 - 4a_4 = 0 \implies a_4 = \frac{1}{2 \cdot 3 \cdot 4}$$

$$y = 4 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots \frac{1}{n!} x^n$$

$$\boxed{y = 3 + e^x}$$