EXAM II SOLS Sp 05

1. Evaluate the following integrals. You must show your work.

(a) $\displaystyle\int_2^8 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx = \int_{\sqrt{2}+1}^{\sqrt{8}+1} 2\frac{du}{u} = 2\ln|u|\Big|_{\sqrt{2}+1}^{\sqrt{8}+1}$

$u = \sqrt{x}+1$

$du = \frac{1}{2\sqrt{x}}dx$

$2du = \frac{1}{\sqrt{x}}dx$

$2 \leq x \leq 8$

$\sqrt{2}+1 \leq \sqrt{x}+1 \leq \sqrt{8}+1$

$= \boxed{2\left[\ln|\sqrt{8}+1| - \ln|\sqrt{2}+1|\right]}$

$= \boxed{2\ln(3-\sqrt{2})}$

(b) $\displaystyle\int \frac{x^2-x+1}{x^2-x}\,dx$ (Hint: Divide) $= \int 1\,dx + \int \frac{-1}{x}\,dx + \int \frac{1}{x-1}\,dx$

$= \boxed{x - \ln|x| + \ln|x-1| + C}$

$$\begin{array}{r} 1 \\ x^2-x \,\overline{)\, x^2-x+1} \\ \underline{x^2-x} \\ 1 \end{array}$$

So $\dfrac{x^2-x+1}{x^2-x} = 1 + \dfrac{1}{x^2-x}$

$\dfrac{1}{x^2-x} = \dfrac{1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

$1 = A(x-1) + Bx$

$1 = Ax - A + Bx$

$1 = -A \implies A = -1$

$0 = A + B \qquad B = 1$

(c) $\displaystyle\int \frac{x^3+1}{x^2+1}\,dx$ (Hint: Divide) $= \int x\,dx + \int \frac{1}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$

$= \boxed{\frac{x^2}{2} + \tan^{-1}x - \frac{1}{2}\ln|x^2+1| + C}$

$$\begin{array}{r} x \\ x^2+1 \,\overline{)\, x^3 \quad +1} \\ \underline{x^3 \quad +x} \\ 1-x \end{array}$$

so $\dfrac{x^3+1}{x^2+1} = x + \dfrac{1-x}{x^2+1}$

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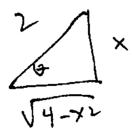
(d) $\displaystyle\int_0^1 \frac{1}{(4-x^2)^{3/2}}\,dx = \int_0^1 \frac{dx}{\left(\sqrt{4-x^2}\right)^3} = \int_0^{\pi/6} \frac{2\cos\theta\, d\theta}{8\cos^3\theta}$




$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = 2\cos\theta$

$0 \le x \le 1$

$0 \le \sin\theta \le 1/2$

$0 \le \theta \le \pi/6$

$$= \frac{1}{4}\int_0^{\pi/6} \frac{1}{\cos^2\theta}\, d\theta = \frac{1}{4}\int_0^{\pi/6} \sec^2\theta\, d\theta$$

$$= \frac{1}{4}\tan\theta \Big|_0^{\pi/6} = \frac{1}{4}\left[\tan\frac{\pi}{6} - \tan 0\right]$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{3}} - 0\right] = \boxed{\frac{1}{4\sqrt{3}}}$$

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(e) $\displaystyle\int_0^1 \frac{1}{x^{1/3}}\,dx =$

Improper integral. ( function not defined at $x=0$ ).

$$\int_0^1 \frac{1}{x^{1/3}}\,dx = \lim_{a\to 0^+} \int_a^1 x^{-1/3}\,dx = \lim_{a\to 0^+} \left[\frac{x^{2/3}}{2/3}\right]\Bigg|_a^1$$

$$= \lim_{a\to 0^+} \frac{3}{2}\left[1^{2/3} - a^{2/3}\right]$$

$$= \frac{3}{2}\lim_{a\to 0^+}\left[1 - a^{2/3}\right] = \frac{3}{2}[1-0] = \boxed{\frac{3}{2}}$$

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(f) $\displaystyle\int t\sin 3t\,dt = -\frac{1}{3}t\cos 3t - \int -\frac{1}{3}\cos 3t$

Integrate by parts.

$u = t \qquad du = dt$

$v = -\frac{1}{3}\cos \qquad dv = \sin 3t\, dt$

$\int u\,dv = uv - \int v\,du$

$w = 3t$

$dw = 3\,dt$

$\frac{1}{3}dw = dt$

$$= -\frac{1}{3}t\cos 3t + \frac{1}{3}\int \cos w\, dw$$

$$= \boxed{-\frac{1}{3}t\cos 3t + \frac{1}{9}\sin 3t + C}$$

2. Determine whether each sequence converges or diverges. If it converges, find the limit. If it diverges, give some explanation (short) why.

(a) $a_n = \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{2}\right)\left(1-\frac{1}{n}\right)$$

$$= \left(\frac{1+0}{2}\right)(1-0) = \frac{1}{2}$$

(b) $a_n = \frac{\ln(n+1)}{n^2}$

Consider $\lim_{x\to\infty} \frac{\ln(x+1)}{x^2} \overset{L.H.}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{2x} = \lim_{x\to\infty} \frac{1}{2x(x+1)} = 0$

$\left(\frac{\infty}{\infty}\right)$

$\therefore \boxed{a_n \to 0}$

(c) $a_n = \frac{n\cos n\pi}{n+1} = \frac{n}{n+1} \cdot \cos n\pi.$ (Various answers on this one)

note. $\cos n\pi = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$

also note $\frac{n}{n+1} \to 1$

$\therefore$ as $n\to\infty$ $a_n$ $\begin{cases} \text{gets close to } -1 & \text{if } n \text{ is odd} \\ \text{gets close to } 1 & \text{if } n \text{ is ev} \end{cases}$

$\therefore$ $\boxed{\lim_{n\to\infty} a_n \text{ does not exist. (diverges)}}$

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3. Consider the function $f(x) = 1/(x+1)$. We compute the 3rd order Taylor polynomial about $a = 2$ by writing the following table.

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(2)$ |
|---|---|---|
| 0 | $(x+1)^{-1}$ | $3^{-1}$ |
| 1 | $-(x+1)^{-2}$ | $-3^{-2}$ |
| 2 | $2(x+1)^{-3}$ | $2!3^{-3}$ |
| 3 | $-3!(x+1)^{-4}$ | $-3!3^{-4}$ |
| 4 | $4!(x+1)^{-5}$ | $4!3^{-5}$ |
| 5 | $-5!(x+1)^{-6}$ | $-5!3^{-6}$ |

We get $T_3(x) = \frac{1}{3} - \frac{1}{9}x + \frac{1}{27}x^2 - \frac{1}{81}x^3$. Use the Taylor Inequality to bound the error due to the approximation of f(x)=1/(x+1) by $T_3$ on the interval [1.5, 2.5]. (Since you do not have calculators, you do not have to do the arithmetic.)

$$|R_3(x)| \leq \frac{M}{4!}|x-2|^4$$

when $M$ is $\max\{|f^{(4)}(x)|\}$ on $[1.5, 2.5]$

Find $M$

$f^{(4)}(x) = \dfrac{4!}{(x+1)^5}$ is positive & decreasing on $[1.5, 2.5]$.

$$\max f^4(x) = f^{(4)}(1.5) = \frac{4!}{(1.5+1)^5} = \frac{4!}{2.5^5}$$

$$\& \quad |R_3(x)| \leq \frac{4!}{2.5^5}\,\frac{|1.5-2|^4}{4!} = \frac{.5^4}{2.5^5} = \boxed{2.5^{-5}(.5)^4}$$

algebraically: $\left(\frac{2}{5}\right)^5 \cdot \left(\frac{1}{2}\right)^4 = \frac{2}{5^5}$ = very small.

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4. Let $g(x) = -6x^4 + 3x^3 - 17x^2 + 2x + 5$ and let $T_n(x)$ be the $n$th degree Taylor polynomial of $g(x)$ at $a = 0$.
(a) What is $T_7(x) - T_5(x)$? Justify your answer.

$$T_7 = T_4 + \frac{^5(0)}{5!}x^5 + \frac{^6(0)}{6!}x^6 + \frac{^7(0)}{7!}x^7$$

$$T_5 = T_4 + \frac{^5(0)}{5!}x^5$$

| $n$ | $g^n(x)$ |
|---|---|
| 0 | $g(x)$ |
| 1 | $-24x^3 +$ |
| 2 | $-72x^2 +$ |
| 3 | $-144x +$ |
| 4 | $-144$ |
| 5 | 0 |
| 6 | 0 |

$T_7(x) = T_4(x)$

$T_5(x) = T_4(x)$ $\quad \therefore \; T_7(x) - T_5(x) = 0$

(b) Find the remainder term $R_3(x)$. Justify your answer

$$T_3(x) = 5 + 2x + \frac{-34x^2}{2!} + \frac{18x^3}{3!}$$

$$= 5 + 2x + 3x^3$$

$$R_3(x) = \quad - T_3(x) = \boxed{-6x^4}$$

5. Find the 3rd order Taylor polynomial of $f(x) = e^{3x}$ centered at $a = 0$.

| $n$ | $f^n(x)$ | $f^n(0)$ |
|---|---|---|
| 0 | $e^{3x}$ | 1 |
| 1 | $3e^{3x}$ | 3 |
| 2 | $9e^{3x}$ | 9 |
| 3 | $27e^{3x}$ | 27 |

$$T_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f^2(0)}{2!}x^2 + \frac{f^3(0)}{3!}$$

$$= 1 + \frac{3}{1!}x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3$$

$$\boxed{T_3(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3}$$

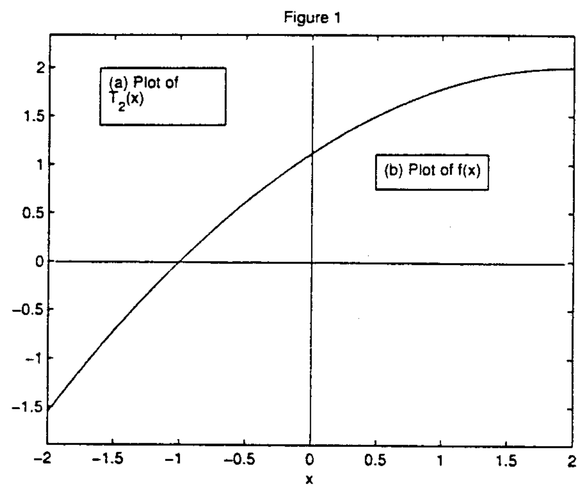
6. (a) Assume that the 2nd order Taylor polynomial of the function $f$ about $a = 0$ is given by $T_2(x) = a + bx + cx^2$. What can you say about the signs of $a$, $b$ and $c$ if the graph of $T_2(x)$ is given below in Figure 1? Explain.

$T_2(x) = a + bx + cx^2.$

$T_2(0) = a = 1 \quad a > 0$

$T_2'(0) = b > 0$ since the curve has positive slope

$T_3''(0) = 2c < 0$ since curve is concave downward.




(b) What can you say about the signs of $a$, $b$ and $c$ if the graph of $f(x)$ is given above in Figure 1? Explain.

$f(x) = ax + bx + cx^2$

$a > 0$ since $f(0) = a = 1$

$b > 0$ since $f'(0) = b$ + curve has pos. slope

$c < 0$ since $f''(0) = 2c$ + curve is concave downward

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7. Let $R_n(x)$ be the $n$th order remainder term for the Taylor polynomial of the function $f$ expanded about $a = 0$. Assume that the remainder term $R_2(x)$ is given by

$$R_2(x) = \frac{1}{2!}\int_a^x f^{(3)}(t)(x-t)^2\,dt.$$

Use integration by parts to show that $R_3(x) = \frac{1}{3!}\int_a^x f^{(4)}(t)(x-t)^3\,dt.$

$$R\ \frac{1}{2!}\int_a^x f^3(t)(x-t)^2\,dt$$

$F = (x-t)^2$
$G = f^3(t)$
$G' = f^4(t)\,dt$
$F = -\frac{(x-t)^3}{3}$

$$= \frac{1}{2!}\left[-\frac{(x-t)^3}{3} f^3(t)\right]_a^x - \frac{1}{2!}\int_a^x -\frac{(x-t)^3}{3} f^4(t)\,dt.$$

$$= -\frac{1}{3!}\left[(x-x)^3 f^3(x) - (x-a)^3 f^3(a)\right] + \frac{1}{3!}\int_a^x (x-t)^3 f^4(t)\,dt$$

$$= -\frac{1}{3!}\left[0 - (x-a)^3 f^3(a)\right] + \frac{1}{3!}\int_a^x (x-t)^3 f^4(t)\,dt.$$

$$= \underbrace{\frac{1}{3!}(x-a)^3 f^3(a)}_{\text{this becomes the next term in } T_3(x)} + \underbrace{\frac{1}{3!}\int_a^x (x-t)^3 f^4(t)\,dt}_{= R_3(x)}.$$

8. Use the Nondecreasing Sequence Theorem [The Nondecreasing Sequence Theorem: A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.] to prove that the sequence $\left\{\frac{2n+1}{3n+4}\right\}$ converges. Also use the Theorem to prove that the sequence converges to 2/3.

(i) show $a_n$ increasing.

$a_{n+1} \overset{?}{\geq} a_n \iff \frac{2(n+1)+1}{3(n+1)+4} \geq \frac{2n+1}{3n+4} \iff \frac{2n+3}{3n+7} \geq \frac{2n+1}{3n+4}$

Actually much easier to consider the fct $f(x) = \frac{2x+1}{3x+4}$ & show this is decreasing (on $[1,\infty)$) (it could be done by looking at $f'(x)$)

$\iff (2n+3)(3n+7) \geq (2n+1)(3n+4)$

$\iff 6n^2 + 17n + 12 \geq 6n^2 + 17n + 7$

$\iff 12 \geq 7$ true. $\therefore a_{n+1} \geq a_n \ \forall n.$

$\therefore$ the sequence is non decreasing.

(ii) show the seq. is bounded (by 2/3)

$a_n \leq 2/3 \iff \frac{2n+1}{3n+4} \leq \frac{2}{3} \ \forall n$

$\iff (2n+1)\cdot 3 \leq (3n+4)\cdot 2 \iff 6n+3 \leq 6n+4$ true since $3 \leq 4$

$\therefore a_n \leq 2/3 \ \forall n$

Since $a_n$ is non decreasing & bounded from above by 2/3 $a_n$ converges

(iii) $a_n$ converges to

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{3n+4} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{3+\frac{4}{n}} = \boxed{\frac{2}{3}}$