Solutions to Practice Exam 1 spring 05

1. (a) Use the properties of logarithms to simplfy $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right)$.

$\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right)$

$= \ln(3x(x-3)) + \ln(1) - \ln(3x)$

$= \ln \cancel{3x} - \ln(x-3) + \ln(1) - \ln(\cancel{3x})$

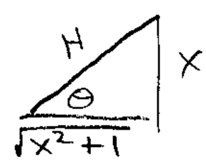
$= +\ln(x-3) + 0$

$= -\ln(x-3) = \ln\left(\frac{1}{x-3}\right)$

(b) Evaluate the expression $\sin\left(\arctan\left(\frac{x}{\sqrt{x^2+1}}\right)\right)$.

$\sin(\arctan \frac{x}{\sqrt{x^2+1}})$

$\arctan \frac{x}{\sqrt{x^2+1}} = \theta \quad \tan\theta = \frac{x}{\sqrt{x^2+1}} = \frac{O}{A}$



$H = \sqrt{x^2 + \sqrt{x^2+1}^2}$

$H = \sqrt{x^2 + x^2 + 1}$

$= \sqrt{2x^2+1}$

$\sin(\arctan \frac{x}{\sqrt{x^2+1}}) = \sin\theta = \frac{x}{\sqrt{2x^2+1}}$

(c) Solve for y in terms of x. $\ln(y^2 - 1) - \ln(y-1) = \ln(\sin x)$

$\ln(y^2-1) - \ln(y-1) = \ln(\sin x)$

$\ln((y+1)(y-1)) - \ln(y-1) = \ln \sin x$

$\ln(y+1) + \cancel{\ln(y-1)} - \cancel{\ln(y-1)} = \ln \sin x$

$\ln y+1 = \ln \sin x$

$e^{\ln y+1} = e^{\ln \sin x}$

$y+1 = \sin x$
$-1 \qquad -1$

$y = \sin x - 1$

2. Calculate the following derivatives (you do not have to simplify).

(a) $\frac{d}{dx} \ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}} = \frac{d}{dx} \ln \left(\frac{(x+1)^5}{(x+2)^{20}} \right)^{1/2}$

$= \frac{d}{dx} \frac{1}{2} \ln \left(\frac{(x+1)^5}{(x+2)^{20}} \right) = \frac{1}{2} \frac{d}{dx} \left(\ln (x+1)^5 - \ln (x+2)^{20} \right)$

$= \frac{1}{2} \frac{d}{dx} \left(5 \ln(x+1) - 20 \ln(x+2) \right)$

$= \frac{1}{2} \left(5 \frac{d}{dx} \ln(x+1) - 20 \frac{d}{dx} (\ln(x+2)) \right)$

$= \frac{1}{2} \left(5 \cdot \frac{1}{x+1} - 20 \cdot \frac{1}{x+2} \right) = \underline{\underline{\frac{1}{2} \left(\frac{5}{x+1} - \frac{20}{x+2} \right)}}$

(b) $\frac{d}{dx} e^{4\sqrt{x}+x^2}$

let $u = 4\sqrt{x} + x^2$

$\frac{du}{dx} = 4 \cdot \frac{1}{2} \cdot x^{-1/2} + 2x$

$\frac{du}{dx} = \frac{2}{\sqrt{x}} + 2x$

$= \frac{d}{dx} e^u$

$= e^u \frac{du}{dx}$

$= \underline{\underline{\left(e^{4\sqrt{x}+x^2} \right) \left(\frac{2}{\sqrt{x}} + 2x \right)}}$

(c) $\frac{d}{dx} \tan^{-1}(\ln x)$

$\frac{d}{dx} \tan^{-1} u = \frac{du/dx}{1+u^2}$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$\frac{d}{dx} \tan^{-1} u = \frac{du/dx}{1+u^2} = \frac{1/x}{1+(\ln x)^2} = \underline{\underline{\frac{1}{x(1+(\ln x)^2)}}}$

3. Evaluate the following integrals. You must show your work.

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx$

if $u = \frac{x}{2}$, $du = \frac{dx}{2}$,

then $\displaystyle\int_0^1 \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx = \int_0^1 \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$

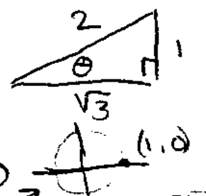
$\displaystyle= \int_{x=0}^{x=1} \frac{1}{\sqrt{1+u^2}} \cdot 2\,du$

$\displaystyle= 2\int_{x=0}^{x=1} \frac{du}{\sqrt{1-u^2}} = 2\left(\sin^{-1} u\right)\Big|_{x=0}^{x=1}$

$\sin^{-1}\left(\frac{1}{2}\right) = \theta$

$\theta = \pi/6$

$\sin^{-1}(0) = 0$



$\displaystyle= 2\sin^{-1}\left(\frac{x}{2}\right)\Big|_0^1 = 2\left(\sin^{-1}\tfrac{1}{2} - \sin^{-1} 0\right)$

$= 2(\pi/6 - 0)$

$= \pi/3$

(b) $\displaystyle\int \frac{1}{3x-2}\,dx$

let $u = 3x-2$

$du = 3dx \quad dx = \frac{du}{3}$

$\displaystyle\int \frac{1}{3x-2}\,dx = \int \frac{1}{u}\,\frac{du}{3} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C$

(c) $\displaystyle\int \frac{e^x + e^{-x}}{\cosh x}\, dx$ $\qquad \cosh x = \frac{e^x + e^{-x}}{2}$

$$\int \frac{e^x + e^{-x}}{\cosh x}\, dx = \int \frac{e^x + e^{-x}}{\frac{e^x + e^{-x}}{2}}\, dx = \int \frac{2(e^x + e^{-x})}{e^x + e^{-x}}\, dx$$

$$= \int 2\, dx = \underline{\underline{2x + C}}$$

(d) $\displaystyle\int_{\ln 2}^{\ln 3} e^x\, dx$ (Simplify your answer.)

$$\int_{\ln 2}^{\ln 3} e^x dx = e^x \Big|_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3 - 2 = \underline{\underline{1}}$$

(e) $\displaystyle\int \cosh(2x)\, dx$ $\qquad$ let $u = 2x$. $du = 2dx$, $dx = \frac{du}{2}$

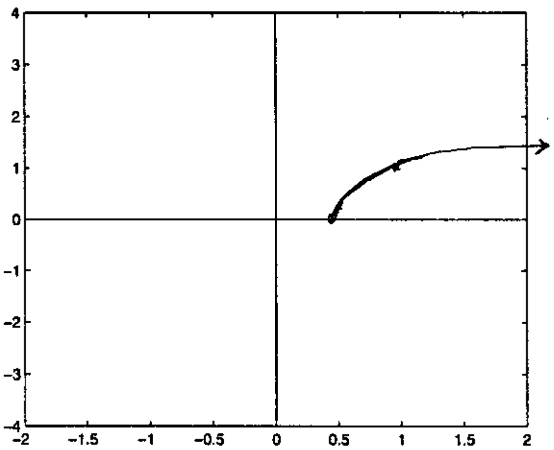
$$= \int \cosh(u) \frac{du}{2} = \frac{1}{2}\int \cosh(u)du = \frac{1}{2} \cdot \sinh u + C = \boxed{\frac{1}{2}\sinh(2x) + C}$$

4. Consider the function $f(x) = \sqrt{\ln(x) + 1}$.

(a) Find the domain and range of $f(x)$.

domain: need $\sqrt{\ln(x)+1}$ to make sense, so need $\ln(x)+1 \geq 0$.
Then $\ln(x) \geq -1$, $e^{\ln x} \geq e^{-1}$, $x \geq e^{-1}$. (need $\ln x$ to make sense also, but $\frac{1}{e} > 0$ so this is o.k.)
domain is $[\frac{1}{e}, \infty)$
Set domain so $\ln(x)+1 \geq 0$, so we obtain $\sqrt{\ln(x)+1} \geq 0$! so the range is $[0, \infty)$

(b) Plot f on the axis below.



x	$f(x)$
$1/2.7 \approx 1/e$	$\sqrt{0} = 0$
1	$\sqrt{1} = 1$
$2.7 \approx e$	$\sqrt{2} \approx 1.4$
2	$\sqrt{\ln(2)+1} \approx 1.3$

(c) Give a short explanation why you know that f will have an inverse. f is 1 to 1, because it passes the horizontal line test, all 1:1 functions have inverses.

(d) Find $f^{-1}(x)$.

$y = \sqrt{\ln x + 1}$ $\quad y^2 = \ln x + 1$ $\quad y^2 - 1 = \ln x$

$e^{y^2-1} = x$. $\quad y \geq 0$, for $f(x)$,

$f^{-1}(x) = y = e^{x^2-1}$ $\quad$ so $x \geq 0$ for $f^{-1}(x)$

(e) Find the domain and range of $f^{-1}(x)$.

domain: $[0, \infty)$. range = domain of f = $[1/e, \infty)$

5. Calculate the following limits.

(a) $\lim_{x\to 0+} x\cot x = \lim_{x\to 0^+} \frac{x}{\tan x} = \lim_{x\to 0^+} \frac{1}{\sec^2 x} = \underline{\underline{1}}$

l'hôpital's rule

(b) $\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \lim_{x\to 0} \frac{\sin x}{1+2x} = \underline{\underline{0}}$

l'hôpital's rule

(c) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{2(1/2(x^{-1/2}))} = \lim_{x\to\infty} \frac{1/x}{x^{-1/2}} =$

l'hôpital's rule

$= \lim_{x\to\infty} \frac{x^{1/2}}{x} = \lim_{x\to\infty} \frac{1/2\, x^{-1/2}}{1} = \lim_{x\to\infty} \frac{1}{2x^{1/2}} = \underline{\underline{0}}$

l'hôpital's rule

6. Which of the following functions grows faster or slower than x^2 (or do they grow at the same speed) as x approaches infinity? Explain.

(a) $\sqrt{x^4 + x^3}$

$$\lim_{x\to\infty} \frac{\sqrt{x^4+x^3}}{x^2} = \lim_{x\to\infty} \frac{\sqrt{x^4(1+1/x)}}{x^2}$$

$$= \lim_{x\to\infty} \frac{x^2\sqrt{1+1/x}}{x^2} = \lim_{x\to\infty} \sqrt{1+1/x} = \underline{\underline{1}}$$

So x^2 and $\sqrt{x^2+x^3}$ are about the same size as x grows very large, and that they grow at the same rate.

(b) $x^2 e^{-x}$

$$\lim_{x\to\infty} \frac{x^2 e^{-x}}{x^2} = \lim_{x\to\infty} e^{-x} = 0,$$

meaning that x^2 is much, much larger than $x^2 e^{-x}$ as x gets very large,

and that x^2 grows faster than $x^2 e^{-x}$

7. Find the solution to the differential equation $\dfrac{dy}{dx} = \dfrac{2xy+2x}{x^2-1}$. Write your solution as a function of x.

$$\frac{dy}{dx} = \frac{2xy+2x}{x^2-1} = \frac{2x(y+1)}{x^2-1}$$

① separate variables

$$\frac{dy}{y+1} = \frac{2x}{x^2-1} \cdot dx$$

② integrate

$$\int \frac{dy}{y+1} = \int \frac{2x}{x^2-1}\, dx$$

$u = x^2 - 1$
$du = 2x\,dx \ldots$

$$\ln|y+1| + C_1 = \ln|x^2-1| + C_2$$

③ solve for y

$$\ln|y+1| = \ln|x^2-1| + C$$

$$e^{\ln|y+1|} = e^{\ln|x^2-1|+C}$$

$$|y+1| = e^{\ln|x^2-1|} \cdot e^C$$

but $\pm$ works for $|x^2-1|$, too, so

$$y+1 = \pm e^C \cdot |x^2-1|$$

$$y+1 = \pm e^C (x^2-1)$$

$$\underline{y = -1 \pm e^C (x^2-1)}$$