M161, Test 3, Fall 2004

NAME:______________________________

SECTION: __________________________

INSTRUCTOR: ______________________

You may use calculators. No formulas can be stored in your calculators.

Problem	Points	Score
1abc	15	
1d	5	
2	10	
3	10	
4	15	
5	15	
6	15	
7	10	
8	5	
Total	100	

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Error Bounds. Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T is the error in the Trapezoidal Rule, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \text{ or it may be written as } |E_T| \leq \frac{K\Delta x^2(b-a)}{12}.$$

Suppose $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_S is the error involved in using Simpson's Rules, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} \text{ or it may be written as } |E_S| \leq \frac{K\Delta x^4(b-a)}{180}.$$

(d) $\sum_{n=4}^{\infty} \frac{3^{2n}}{n^5}$

root test

$\sqrt[n]{a_n} = \frac{9}{(\sqrt[n]{n})^5}$

$\rho = \lim \sqrt[n]{a_n} = 9 > 1$

divergent.

2. Determine whether the series is absolutely convergent, conditionally convergent or divergent. In any case, **explain why**.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$

$\sum \frac{\ln n}{n} > \sum \frac{1}{n}$ which diverges

AST $\sum (-1)^n \frac{\ln n}{n}$ converges

conditionally convergent.

$f(x) = \frac{\ln x}{x}$ $\quad f'(x) = \frac{x \frac{1}{x} - \ln x}{x^2}$

(b) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$

ratio test $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim \frac{100}{(n+1)!} \cdot n! = \lim \frac{100}{n+1} = 0$

absolutely convergent.

3. Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x+4)^n}{n3^n}$.

root test $\sqrt[n]{|a_n|} = \frac{|x+4|}{\sqrt[n]{n} \cdot 3}$

$\rho = \frac{|x+4|}{3} < 1$

center -4 radius 3

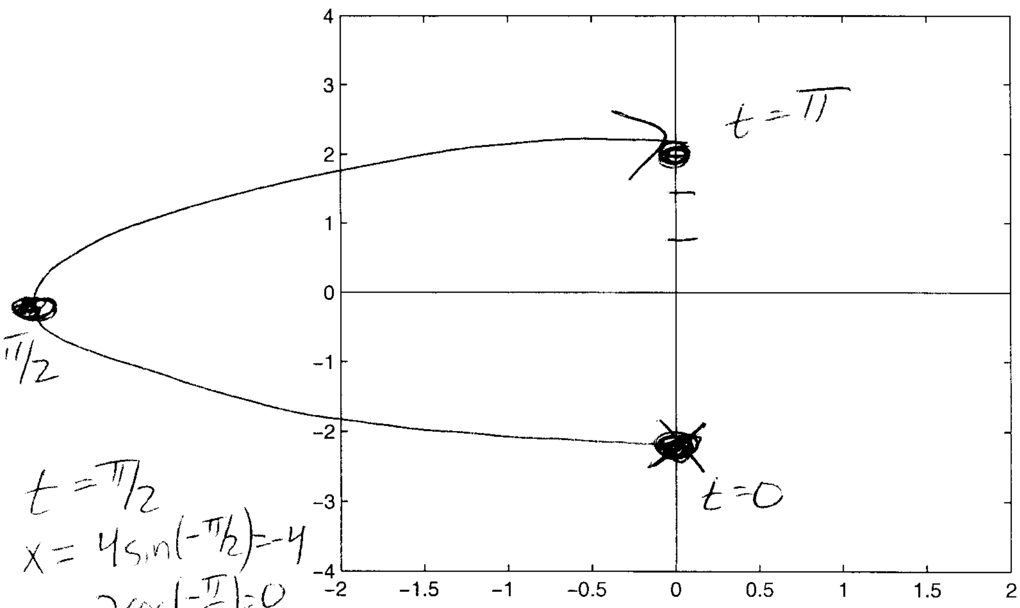
endpoints $x = -7$ $\sum \frac{(-1)^n}{n}$ converges

$x = -1$ $\sum \frac{1}{n}$ diverges

$[-7, -1)$

4. The parametric equations $x = 4\sin(t-\pi)$, $y = 2\cos(t-\pi)$, $0 \le t \le \pi$ describe the motion of a particle in the x-y plane.

(a) Sketch the path of the particle indicating the direction of motion.



$t = \pi/2$

$t = \pi/2$
$x = 4\sin(-\pi/2) = -4$
$y = 2\cos(-\pi/2) = 0$

$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

$t = 0$
$x = 4\sin(-\pi) = 0$
$y = 2\cos(-\pi) = -2$

$t = \pi$
$x = 4\sin 0 = 0$
$y = 2\cos 0 = 2$

(b) What is the equation of the tangent to the curve at the point where $t = 3\pi/4$? Draw the tangent line on the curve drawn above.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin(t-\pi)}{4\cos(t-\pi)}$$

$$\left.\frac{dy}{dx}\right|_{t=3\pi/4} = \frac{-2\sin(-\pi/4)}{4\cos(-\pi/4)} = \frac{2\sqrt{2}/2}{4\sqrt{2}/2} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

$t = 3\pi/4$ $\quad x = 4\sin(-\pi/4) = -4\frac{\sqrt{2}}{2} = -2\sqrt{2}$ $\quad y = 2\cos(-\pi/4) = 2\frac{\sqrt{2}}{2} = \sqrt{2}$

$$\boxed{y - \sqrt{2} = \frac{1}{2}(x + 2\sqrt{2})}$$

$$\boxed{y = \frac{1}{2}x + (2\sqrt{2})}$$

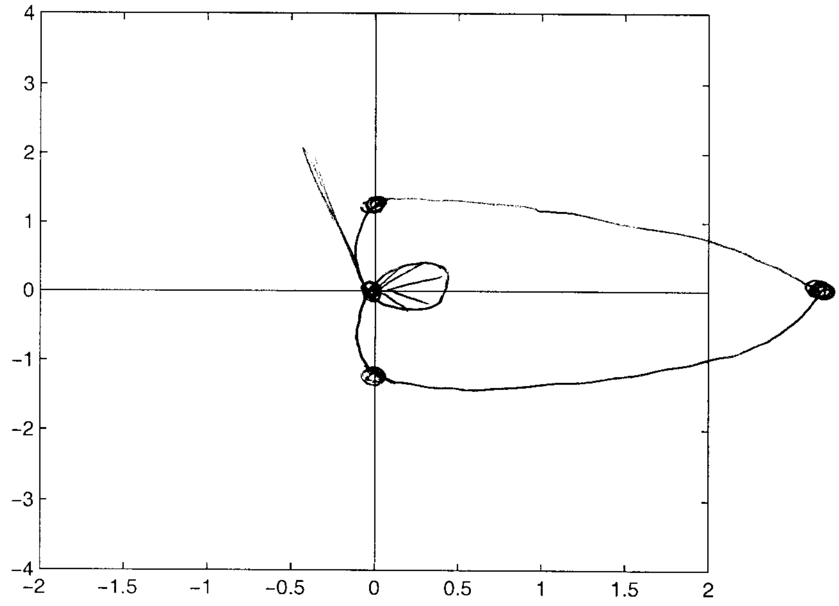
(c) What are the Cartesian coordinates of the point where $t = \pi$.

$(0, 2)$

(d) Find the Cartesian equation for the parametric equations.

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

5. (a) Sketch the curve $r = 1 + 2\cos\theta$.



$r = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}$

$\theta = \frac{4\pi}{4}$

(b) Find the area inside the large loop of the curve.

$$A = 2\cancel{4} \int_0^{2\pi/3} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi/3} \left(1 + 4\cos\theta + 4\cos^2\theta\right) d\theta$$

$$= \left[\theta + 4\sin\theta + 2\int 1 + \cos(2\theta)\right]$$

$$= \left[3\theta + 4\sin\theta + \sin(2\theta)\right]_0^{2\pi/3} = 2\pi + 4\frac{\sqrt{3}}{2} + \frac{-\sqrt{3}}{2} - 0 - 0 - 0$$

$$= \boxed{2\pi + 3\sqrt{3}/2}$$

(c) Find the area inside the small loop of the curve.

$$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2}\left(r^2\right) d\theta = \left[3\theta + 4\sin\theta + \sin(2\theta)\right]\Big|_{2\pi/3}^{\pi}$$

same integral

$$= \left(3\pi + \cancel{4\tfrac{\sqrt{3}}{2}}\, 0 + 0\right) - \left(2\pi + 4\frac{\sqrt{3}}{2} + \frac{-\sqrt{3}}{2}\right)\Big]$$

$$= \boxed{\pi - 3\frac{\sqrt{3}}{2}}$$

6. (a) Find the Cartesian equation form of the polar curve $r\cos\theta - y\sin\theta = 7$.

r

$$\boxed{x - y = 7}$$

$\sin\theta = \frac{y}{r}$

$= \cancel{\frac{y}{\sqrt{x^2+y^2}}}$

(b) Find the polar equation for the Cartesian equation $xy = 2$.

$$r^2\cos\theta\sin\theta = 2$$

$$\boxed{r^2 = \frac{2}{\cos\theta\sin\theta}}$$

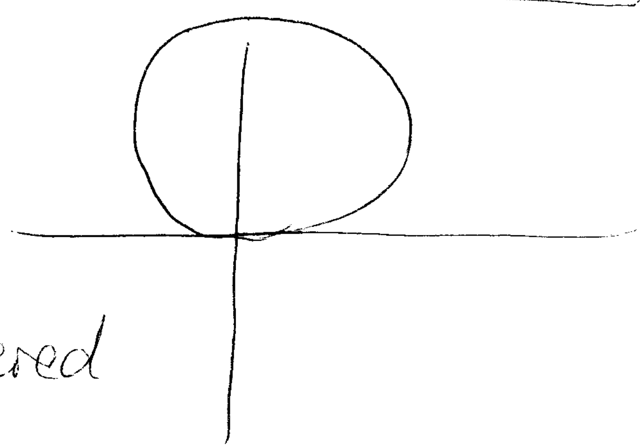
(c) Find the Cartesian equation form of the polar curve $r^2 = 4r\sin\theta$. Identify the curve.

$\cancel{r^2 = 4r\sin\theta}$

$$\boxed{r = 0 \quad \text{or} \quad r = 4\sin\theta}$$

$$x^2 + y^2 = 4y$$

$$\boxed{x^2 + (y-2)^2 = 4}$$



~~ce~~ circle centered at $(0,2)$ radius 2

7. Use power series to solve the initial value problem $y' + 2y = 0$, $y(0) = 1$.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$y' + 2y = (a_1 + 2a_0) + (2a_2 + 2a_1)\cancel{x} + (3a_3 + 2a_2)x^2 + \ldots (na_n + 2a_{n-1})x^{n-1}$$

$y(0) = 1 \Rightarrow \boxed{a_0 = 1}$

$a_1 + 2a_0 = 0 \Rightarrow \boxed{a_1 = -2}$

$2a_2 + 2a_1 = 0 \Rightarrow \boxed{a_2 = +2}$

$3a_3 + 2a_2 = 0 \quad \boxed{a_3 = -4/3}$

$$y = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!} x^n$$

$$\frac{dy}{y} = -2dx$$

$$\ln|y| = -2x$$

$$\boxed{y = \cancel{A} e^{-2x}}$$

8. Find the power series representation of $f(x) = \frac{1}{(1-x)^3}$. Hint: $\frac{d^2}{dx^2}\frac{1}{1-x} = \frac{2}{(1-x)^3}$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad x^n$$

$\downarrow d/dx$

$$\frac{+1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots nx^{n-1}$$

$$\frac{2}{(1-x)^3} = 2 + 6x + \ldots n(n-1)x^{n-2}$$

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$