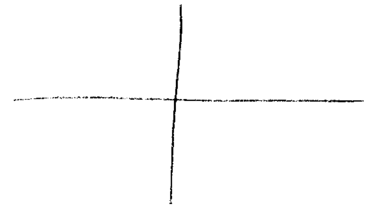
Exam practice: complex numbers + trapezoid rule due 12/1 Mon



1. (a) Simplify $\dfrac{(2+i)(3+i-1)}{1+i-3}$. (Write as $a+bi$, a and b real.)

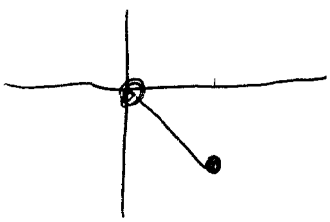
$$= \frac{(2+i)(+2+i)}{(-2+i)} = \frac{(2+i)^3}{4+1} =$$

$$= \boxed{\frac{-2}{5} + \frac{11}{5} i}$$

$(2+i)(2+i) = 4+4i-1$
$= 3+4i$

$(3+4i)(2+i) = 6+11i-4$
$= 2+11i$

(b) Write the complex number $1-i$ in polar form (in the form $re^{i\theta}$, $r>0$ and $-\pi \le \theta \le \pi$).

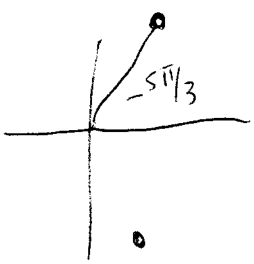


angle $\theta = -\pi/4$

$r = \sqrt{2}$

$$\boxed{z = \sqrt{2}\, e^{-\pi/4\, i}}$$

(c) If the polar form of $\sqrt{3}-i$ is $2e^{-\frac{\pi}{3}i}$, compute $(\sqrt{3}-i)^{23}$. (You may leave your answer in polar form.)

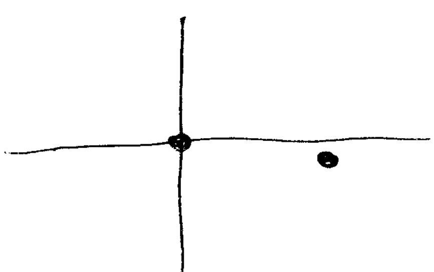


$$\left(2e^{-\frac{\pi}{3}i}\right)^{23} = \left(2^{23}\right) e^{-\frac{23\pi}{3}i} = 2^{23}\left(e^{\left(-\frac{18\pi}{3} - \frac{5\pi}{3}\right)i}\right)$$

$$= 2^{23}\left(e^{-6\pi i}\right) e^{-5\pi/3\, i} = \boxed{2^{23} e^{\pi/3\, i}}$$

$e^{-6\pi i} = 1$

(d) If the polar form of $\sqrt{3}-i$ is $2e^{-\frac{\pi}{3}i}$, compute the fourth roots of $\sqrt{3}-i$. (You may leave your answer in polar form.)



$$z = 2e^{-\frac{\pi}{3}i}$$

$z_1 = z^{1/4} = 2^{1/4} e^{-\pi/12\, i}$ solution 1

$z_2 = z_1 e^{\frac{2\pi i}{4}} = 2^{1/4} e^{5\pi/12}$ sol. 2

$z_3 = z_1 e^{\pi i} = 2^{1/4} e^{\frac{11\pi}{12}}$ sol. 3

$z_4 = z_1 e^{\frac{3\pi i}{2}} = 2^{1/4} e^{\frac{17\pi}{12}}$ sol. 4

$\frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{11}$

$\pi - \frac{\pi}{12} = \frac{11\pi}{12}$

$\frac{3\pi}{2} - \frac{\pi}{12} = \frac{17\pi}{12}$

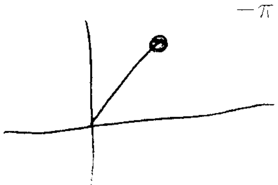
$(2+i)(1+i) = 2+3i-1 = 1+3i$

2 2. (a) Simplify $\frac{2+i}{i-3}(1+i)$.

$$\frac{1}{i-3} = \frac{i+3}{(i-3)(i+3)} = \frac{i+3}{-1-9} = \frac{3+i}{-10}$$

$$(1+3i)\left(\frac{3+i}{-10}\right) = \frac{3+9i+i-3}{-10} = -i$$

(b) Write the complex number $8\sqrt{2}+8\sqrt{2}i$ in polar form, (in the form $re^{i\theta}$, $r>0$ and $-\pi<\theta\leq\pi$).



$r = \sqrt{(8\sqrt{2})^2+(8\sqrt{2})^2}$

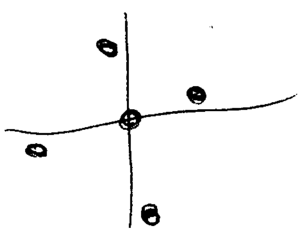
$r = \sqrt{64 \cdot 4}$

$r = 8 \cdot 2 = 16$

$\theta = \frac{\pi}{4}$

$16e^{i\pi/4}$

(c) Find the fourth roots of $8\sqrt{2}+8\sqrt{2}i$. (You may leave your answer in polar form.)



$z_1 = 2e^{i\pi/16}$

$z_2 = 2e^{i9\pi/16}$

$z_3 = 2e^{i17\pi/16}$

$z_4 = 2e^{i25\pi/16}$

$\frac{\pi}{16} + \frac{\pi}{2} =$

(d) Suppose that $z_1 = 2e^{-i\pi/4}$ and $z_2 = 3e^{i\pi/6}$. Compute z_1z_2 and $\frac{z_1}{z_2}$. (You may leave your answer in polar form.)

$$z_1z_2 = 6e^{i(-\pi/4+\pi/6)} = 6e^{-\frac{\pi}{12}i}$$

$$z_1/z_2 = \frac{2}{3}e^{i(-\frac{\pi}{4}-\pi/6)} = \frac{2}{3}e^{-\frac{5\pi}{12}i}$$

2 3. (a) Is $\overline{zw} = \bar{z}\bar{w}$? If true, prove it. If false, find complex numbers z and w for which it is false.

true

$z = a+bi$ $\quad\quad$ $\bar{z} = a-bi$

$w = c+di$ $\quad\quad$ $\bar{w} = c-di$

$zw = (ac-bd)+(bc+ad)i$ $\quad\quad$ $\bar{z}\bar{w} = (ac-bd)-(ad+bc)i$

$\overline{zw} = (ac-bd)-(bc+ad)i$

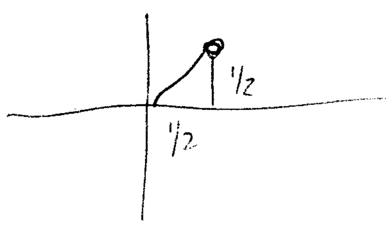
same.

(b) Is $\dfrac{z}{w} = \dfrac{z\bar{w}}{|z|^2}$? If true, prove it. If false, find complex numbers z and w for which it is false.

only if $w\bar{w} = |z|^2$ which is true only if $|w| = |z|$ (same distance from 0 = origin)

false almost always, for example let $z=1$ and $w=2$ then $\dfrac{z}{w} = \dfrac{1}{2} \neq \dfrac{z\bar{w}}{|z|^2} = 2$

(c) Without the use of your calculator, compute $\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}$.



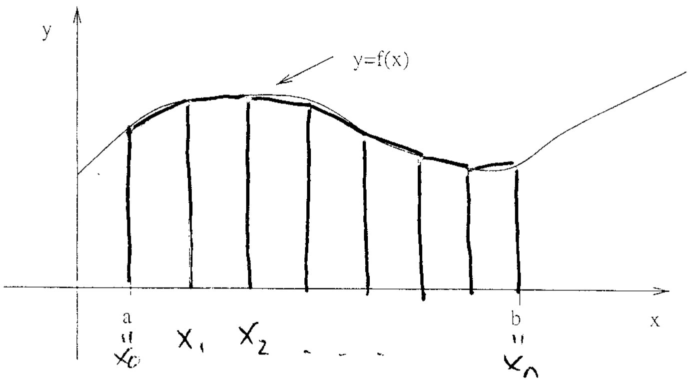
$r = \sqrt{1/2} = 1/\sqrt{2}$ $\quad\quad$ $\theta = \pi/4$

$z = \frac{1}{2}+\frac{1}{2}i = \frac{1}{\sqrt{2}}e^{i\pi/4}$

$z^{10} = \dfrac{1}{2^5}e^{i\frac{10\pi}{4}} = \dfrac{1}{2^5}e^{i\frac{5\pi}{2}}$

$\boxed{z^{10} = \dfrac{1}{2^5}e^{i\pi/2}}$ $\quad$ since $e^{2\pi i} = 1$

2 4. Derive the trapezoidal rule formula for approximating the integral $\int_a^b f(x)\,dx$. Include a drawing on the plot of f given below that illustrates your derivation.



divide $[a,b]$ into n parts of length $\Delta x = \frac{b-a}{n}$

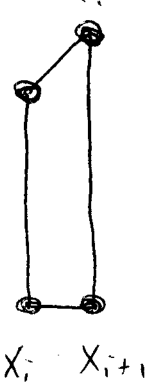
let $x_i = a + \Delta x \cdot i$ $\quad x_0 = a$ and $x_n = b$

let $y_i = f(x_i)$ height of func at x_i

look at trapezoids

~~edge~~ corners at $(x_i, 0), (x_{i+1}, 0)$

(x_i, y_i) & (x_{i+1}, y_{i+1})



Area of trapezoid is

$$\underbrace{(\Delta x)}_{\text{width}}\underbrace{\left(\frac{y_i + y_{i+1}}{2}\right)}_{\text{avege height}}$$

Approximation for integral is the sum of the area of the trapezoids

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n-1} \Delta x \left(\frac{y_i + y_{i+1}}{2}\right)$$