Exam practice : chapter 7 and limits

due 12/5 Mon

6. Calculate the following derivatives (you do not have to simplify).

(a) $\frac{d}{dx}3^{x^2-4}$

$$\boxed{\ln 3\left(3^{x^2-4}\right)(2x)}$$

(b) $\frac{d}{dx}\sin^{-1}(2x)$

$$\boxed{\frac{1}{\sqrt{1\cancel{+}-4x^2}}\cdot 2}$$

(c) $\frac{d}{dx}\sinh(2x)$

$$\sinh(2x) = \frac{e^{2x}-e^{-2x}}{2}$$

$$\frac{d}{dx}(\sinh(2x)) = \frac{2e^{2x}+2e^{-2x}}{2}$$

$$= \boxed{2\cosh(2x)}$$

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(a) $\frac{d}{dx}\sin^{-1}(2x)$

see #1

(b) $\frac{d}{dx}\sinh(2x)$

see #1

7. Solve the following differential equation: $\frac{dy}{dx} = -3y$ (solve for y). Find the solution that satisfies $y(0) = 4$.

$$\frac{dy}{y} = -3dx$$

$$\ln|y| = -3x + C$$

$$\cancel{\ln}\,|y| = e^{-3x+C}$$

$$\boxed{y = Ae^{-3x}}$$

$y(0) = 4 \Rightarrow A = 4$

$$\boxed{y = 4e^{-3x}}$$

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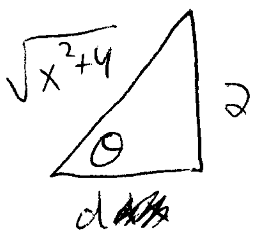
(a) Solve for y in terms of x: $\ln(y^2-1) - \ln(y-1) = \ln(\sin(x))$.

$$e^{\ln(y^2-1)-\ln(y-1)} = e^{\ln(\sin(x))} \Rightarrow \frac{y^2-1}{y-1} = \sin x$$

$$\Rightarrow y+1 = \sin x \Rightarrow \boxed{y = \sin x - 1}$$

(b) Evaluate the expression $\cos(\arcsin(2/\sqrt{x^2+4}))$.



$$\theta = \arcsin\left(\frac{2}{\sqrt{x^2+4}}\right)$$

$$\sin\theta = \frac{2}{\sqrt{x^2+4}}$$

$$\cancel{x}d = \sqrt{(x^2+4)-4} = x$$

$$d = x$$

$$\boxed{\cos(\theta) = \frac{x}{\sqrt{x^2+4}}}$$

(c) For which values of x does the expression in (b) make sense?

domain of arcsin is $[-1, 1]$

need $-1 \le \frac{2}{\sqrt{x^2+4}} \le 1$

need $-\sqrt{x^2+4} \le 2 \le \sqrt{x^2+4}$

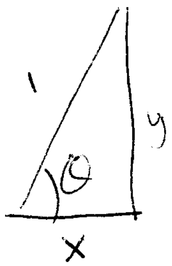
(negative) ↑ always true

need $2 \le \sqrt{x^2+4}$

need $4 \le x^2+4$

need $0 \le x^2$ for all x.

4. (a) Simplify $\cot(\cos^{-1} x)$.



$\theta = \cos^{-1}(x)$

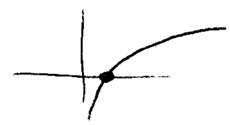
$y = \sqrt{1-x^2}$

$\cot(\theta) = \frac{x}{y} = \boxed{\frac{x}{\sqrt{1-x^2}}}$

(b) Answer true or false.

(i) $\ln 1 = e$. false $\ln(1) = 0$

(ii) The graph of the function $y = \ln x$ is decreasing. false



(iii) The function $f(x) = \ln x$ is not one-to-one. false

(iv) $\lim_{x \to 0+} \ln x = -\infty$ true

(v) The function $f(x) = \sin\theta$, $0 \leq \theta \leq \pi$ is one-to-one. false

(vi) If $f(x) = \sqrt{x-2}$, ≥ 2, then $f^{-1}(x) = \sqrt{x^2+2}$. false $f^{-1}(x) = x^2+2$

(vii) If $f(x) = \sqrt{x-2}$ and $g = f^{-1}$, then $g'(3) = 6$. true

(viii) The function $f(x) = x^2$ is invertible because it satisfies the vertical line test. false

(ix) $\sum_{n=3}^{\infty} \frac{1}{2^n} = 2$

(x) If $\lim_{n \to \infty} a_n = 0$, then the series $\sum_{n=3}^{\infty} a_n$ converges. false

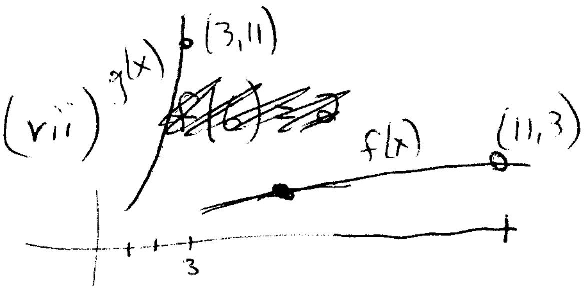
(vi) $y = \sqrt{x-2}$

switch $x = \sqrt{y-2}$

solve for y $x^2 = y-2$

$x^2+2 = y$

(vii)



$\frac{d}{dx}\left(f^{-1}(x)\right)\Big|_{x=a} = \frac{1}{f'(x)}\Big|_{x=b} =$

$g'(3) = \frac{1}{f'(11)}$

$f(x) = 3$

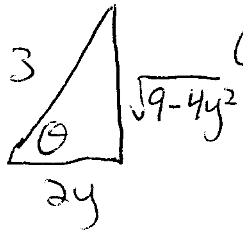
$\sqrt{x-2} = 3$

$x-2 = 9$

$x = 11$

$f'(11) = \frac{1}{2}\frac{1}{\sqrt{x-2}}\Big|_{x=11} = \frac{1}{6}$

5. (a) Simplify $\sin(\cos^{-1} \frac{2y}{3})$.



$\theta = \cos^{-1}\left(\frac{2y}{3}\right)$

$$\sin(\theta) = \frac{\sqrt{9-4y^2}}{3}$$

(b) Determine whether the sequence converge or diverge. If it converges, find the limit. Support your answer with work or an explanation.

(i) $a_n = \frac{3n^2 + n}{2n^2 + n + 1} = \frac{3 + 1/n}{2 + 1/n + 1/n^2}$

$$\lim_{n\to\infty} a_n = \frac{3 + \lim \frac{1}{n}}{2 + \lim \frac{1}{n} + \lim \frac{1}{n^2}} = \frac{3}{2}$$

(ii) $a_n = 2 + (-1)^n$

no limit $\quad a_n = \begin{cases} 3 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases}$

(c) Calculate the limit $L = \lim_{x\to\infty} \frac{2\ln(x)}{\ln(2x)} = \frac{\infty}{\infty}$ use l'hospital

$$L = \lim_{x\to\infty} \frac{2\,\frac{1}{x}}{\frac{1}{2x}\cdot 2} = 2$$

$n \quad 1 \quad 2 \quad 3 \quad 4$

6. (a) Find the formula for the n-th term of the sequence $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}, \cdots$.

$$a_n = (-1)^{n-1}\frac{1}{(2n-1)}$$

(b) Compute $\lim_{n\to\infty} \frac{2n^2+n+1}{3n^2+n}$ if the limit exists. Support your answer with work or an explanation.

answer $\frac{2}{3}$

(c) Compute $\lim_{n\to\infty} \frac{3\ln n}{\ln(2n)}$ if the limit exists. Support your answer with work or an explanation.

answer $= \lim_{x\to\infty} \frac{3\ln x}{\ln(2x)} \overset{\lim}{=} \frac{3\ln x}{\ln(x)+\ln(2)}$

$\overset{\lim}{=} \frac{3}{1+\ln(2)/\ln x} = 3$

(d) Compute $\lim_{x\to\infty} \frac{2\sinh(2x)}{4e^{3x}}$ if the limit exists. Support your answer with work or an explanation.

$$\overset{\lim}{=} \frac{2\left(\frac{e^{2x}-e^{-2x}}{2}\right)}{4e^{3x}} = \lim_{x\to\infty} \frac{e^{2x}-e^{-2x}}{4e^{3x}} =$$

$$\lim_{x\to\infty} \frac{e^{-x}-e^{-5x}}{4} = \boxed{0}$$