Exam practice : Series + Taylor series

due 12/7 Wed

1. Determine whether the series is convergent or divergent. In either case you must justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$ $= \frac{1}{1} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \cdots$

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & n \text{ even} \quad n=2k \\ 1 & n = 1, 5, 9, \ldots \\ -1 & n = 3, 7, 11 \ldots \end{cases}$$

converges by alt. ser. test.

need to show $\left\{\frac{1}{(2k+1)}\right\}$ pos, dec, $\lim = 0$

(b) $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$

converges by root or ratio test.

Show work.

(c) $\sum_{n=1}^{\infty} \frac{n}{(3n^2+n+1)}$

diverges with limit comparison test

$a_n = \frac{n}{3n^2+n+1}$ $\quad b_n = \frac{1}{3n}$

Show $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ $\quad$ Show $\sum b_n$ diverges p-series $p=1$

So $\sum a_n$ diverges also

3. Determine whether the series is convergent or divergent. In either case you must justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+n^2+n+1}$ $b_n = \frac{1}{n^2}$

Show $\lim \frac{a_n}{b_n} = 1$

$\sum b_n$ converges p-series $p=2$

so $\sum a_n$ converges by limit comparison test.

(b) $\sum_{n=1}^{\infty} \frac{n3^n}{(n+1)2^n}$ $b_n = \left(\frac{3}{2}\right)^n$

$\sum b_n$ diverges - geometric series $r = \frac{3}{2} > 1$

$\lim \frac{a_n}{b_n} = 1$

so $\sum a_n$ diverges by limit comparison test.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(3n^2+1)}$

alternating series

$u_n = \frac{n}{3n^2+1}$ Show u_n pos, dec, $\lim = 0$

converges

3. Determine whether the series is absolutely convergent, conditionally convergent or divergent. In any case you must justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+5}}$

conditionally convergent

$\sum \frac{(-1)^n}{\sqrt{n+5}}$ converges by alt. series test

$\sum \frac{1}{\sqrt{n+5}}$ diverges by comparison w/ p-series.

(b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

absolutely convergent

$\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$ & $\sum \frac{1}{n^2}$ converges p-series $p=2$.

(c) $\sum_{n=1}^{\infty} n(n+1)e^n\pi^{-n} = \sum n(n+1)\left(\frac{e}{\pi}\right)^n$

root test $\sqrt[n]{a_n} = \sqrt[n]{n}\,\sqrt[n]{n+1}\,\frac{e}{\pi}$

$\lim \sqrt[n]{a_n} = 1 \cdot 1 \cdot \frac{e}{\pi} < 1$

absolutely converges.

? 4. Determine the power series representation for the function $f(x) = \dfrac{1}{1+9x^2}$ and determine the interval of convergence.

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots = \sum_{n=0}^{\infty} y^n$$

$$\frac{1}{1+9x^2} = 1 - 9x^2 + (-9x^2)^2 + (-9x^2)^3 + \cdots = \sum_{n=0}^{\infty} (-9x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n}$$

root test $\rho = \sqrt[n]{|a_n|} = 9 \cdot |x|^2$ (or geometric series)

need $\rho < 1$

$|x|^2 < \frac{1}{9}$

$|x| < \frac{1}{3}$

center 0 Radius $R = \frac{1}{3}$

endpoints: $x = -\frac{1}{3}$ $\sum_{n=0}^{\infty} (-1)^n \cancel{9^n} \left(\cancel{\frac{1}{9}}\right)^n = \sum (-1)^n$ diverges

$x = \frac{1}{3}$ same

interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$

5. (a) Find the Taylor series expansion of $f(x) = \dfrac{1}{(x-2)^2}$ at $a = 0$. Write the result using summation notation.

method 1: ~~Binomial series~~ plug away w/ formula $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n} x^n$

method 2: $f(x) = \frac{1}{(2-x)^2} = \frac{1}{4(1-\frac{x}{2})^2}$

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$$

deriv.

$$\frac{+1}{(1-y)^2} = \sum_{n=0}^{\infty} n y^{n-1}$$

$$\frac{1}{(1-\frac{x}{2})^2} = \sum_{n=0}^{\infty} n \left(\frac{x}{2}\right)^{n-1}$$

$$\frac{1}{4(1-\frac{x}{2})^2} = \sum_{n=0}^{\infty} \frac{n}{4 \cdot 2^{n-1}} x^{n-1}$$

$$\frac{1}{4(1-\frac{x}{2})^2} = \boxed{\sum_{n=0}^{\infty} \frac{n}{2^{n+1}} x^{n-1} = f(x)}$$

(b) For what values of x does the series ~~found in part (a)~~ $\sum_{n=1}^{\infty} \frac{n}{3^n} x^{n-1}$ converge?

ratio ~~root~~ test $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{3^{n+1}} |x^n| \Big/ \frac{n}{3^n} |x^{n-1}| = \frac{n+1}{n} \frac{1}{3} |x|$

$$\rho = \lim \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}|x|$$

converges if $\rho < 1$ i.e. $|x| < 3$

endpoints: $x = 3$ $\quad \sum \frac{n}{3}$ diverges

$x = -3$ $\quad \sum \frac{(-1)^{n-1} n}{3}$ diverges

so answer $\boxed{-3 < x < 3}$

6. Suppose that f has the following Taylor polynomial expansion centered at $a = 0$: $T_4(x) = 2 + 5x^2 + 4x^4$.

(a) Calculate $f(0)$ and explain your answer.

$f(0) = 2$ constant term in Taylor polynomial

(b) What can you say about the first and third derivatives of f at $x = 0$? Explain your answer (please make the explanation short).

They are both 0 since the first & 3rd Term of Taylor polynomial vanish.

constant term $\frac{f'(0)}{1}x = 0 \Rightarrow f'(0) = 0$

degree 3 term $\frac{f'''(0)}{3!}x^3 = 0 \Rightarrow f'''(0) = 0$.

7 (a) Find the Taylor series expansion of $f(x) = \ln x$ at $x = 2$. Write the result using summation notation.

$f^n(x)$	$f^n(2)$
$\ln x$	$\ln(2)$
$\frac{1}{x}$	$\frac{1}{2}$
$-\frac{1}{x^2}$	$-\frac{1}{2^2}$
$+\frac{2}{x^3}$	$+\frac{2}{2^3}$
$-\frac{6}{x^4}$	$-\frac{6}{2^4}$

$$T(x) = \ln(2) + \frac{1}{2}x - \frac{1}{2}\frac{1}{2^2}x^2 + \frac{1}{3\,2^3}x^3 - \frac{1}{4\cdot 2^4}x^4 + \cdots$$

$$\boxed{= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\cdot 2^n}(x-2)^n}$$

(b) For what values of x does the series found in part (a) converge.

root test $\sqrt[n]{|a_n|} = \frac{|x-2|}{\sqrt[n]{n}\cdot 2}$

$$\rho = \lim \sqrt[n]{|a_n|} = \frac{|x-2|}{2}$$

$$\rho < 1 \iff |x-2| < 2$$

endpoints: $x = 4$ $\ln(2) + \sum \frac{(-1)^{n-1}}{n}$ converges AST

$x = 0$ $\ln(2) + \sum \frac{-1}{n}$ diverges p-series

interval $\cancel{(-2,2)}$ $[0, 4]$

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(a) Find the 3rd degree Taylor polynomial $T_3(x)$ that approximates the function $f(x) = x^{-2}$ about the point $a = 1$.

n	$f^n(x)$	$f^n(1)$
0	$\frac{1}{x^2}$	1
1	$-\frac{2}{x^3}$	-2
2	$\frac{6}{x^4}$	6
3	$-\frac{24}{x^5}$	-24
4	$\frac{5!}{x^6}$	

$$\boxed{T_3(x) = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3}$$

(b) Use the Taylor inequality to estimate the accuracy of the approximation $f(x) \sim T_3(x)$ when x satisfies $.8 \leq x \leq 1.1$.

$$f^4(x) = \frac{5!}{x^6}$$ decreasing when $x > 0$

$M = \max\{|f^4(x)| \; .8 \leq x \leq 1.1\}$ occurs when $x = .8$

$$M = \frac{5!}{(.8)^6}$$

$$|R_3(x)| \leq \frac{M}{4!}|x-1|^4 = \frac{5!}{.8^6 \cdot 4!}|x-1|^4$$

$$|R_3(x)| \leq \frac{5}{(.8)^6}(.2)^4 = \frac{5}{\left(\frac{4}{5}\right)^6}\left(\frac{1}{5}\right)^4 = \frac{5^7}{4^6 \cdot 5^4}$$

$$\boxed{|R_3(x)| \leq \frac{5^3}{4^6}}$$