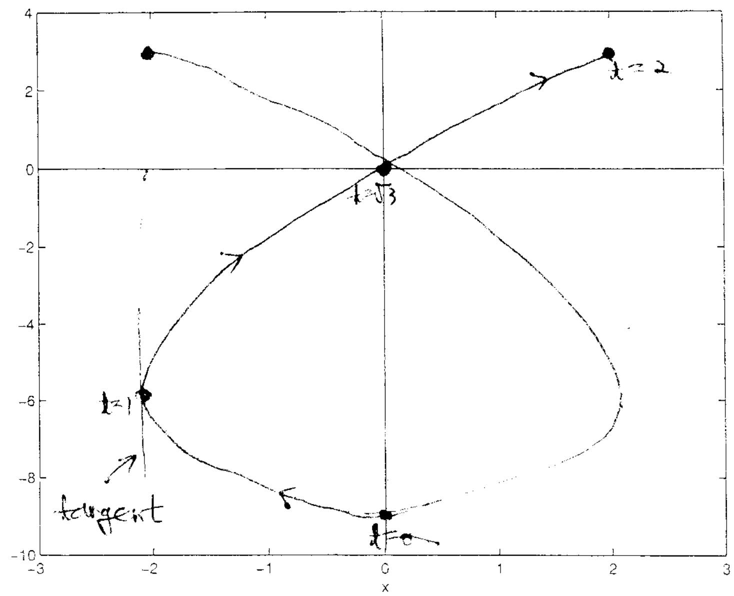
Exam Practice: parametric + polar due 12/9 Fri

1. (a) Sketch the curve of the parametric equations $x = t(t^2 - 3)$, $y = 3(t^2 - 3)$, $-2 \le t \le 2$ and indicate with an arrow the direction in which the curve is traced as the parameter t increases.



trace $x(t)$: odd $x(-t) = -x(t)$

$y(t)$: even $y(-t) = y(t)$.

$0 \le t \le 2$

$x(0) = 0$, $y(0) = 3(-3) = -9$ $\Rightarrow (0,-9)$

$x(1) = 1\cdot(1-3) = -2$, $y(1) = 3(1-3) = -6$ $\Rightarrow (-2,-6)$

$x(\sqrt{3}) = \sqrt{3}\cdot 0 = 0$, $y(\sqrt{3}) = 0$ $\Rightarrow (0,0)$

$x(2) = 2$, $y(2) = 3$ $\Rightarrow (2,3)$

(b) Find the tangent to the curve at the point $(-2, -6)$. Draw the tangent line on your plot of the curve.

$$\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \left.\frac{3\cdot 2t}{3t^2-3}\right|_{t=1} = \infty$$

(c) Find the tangent to the curve at the point $(0, 0)$.

$$\frac{dy}{dx} = \frac{6t}{3t^2-3}$$

$t = \sqrt{3} \Rightarrow (x,y) = (0,0)$, $\left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{6\sqrt{3}}{3\cdot 3 - 3} = \underline{\underline{\sqrt{3}}}$

$t = -\sqrt{3} \Rightarrow (x,y) = (0,0)$, $\left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = \frac{-6\sqrt{3}}{3\cdot 3 - 3} = \underline{\underline{-\sqrt{3}}}$

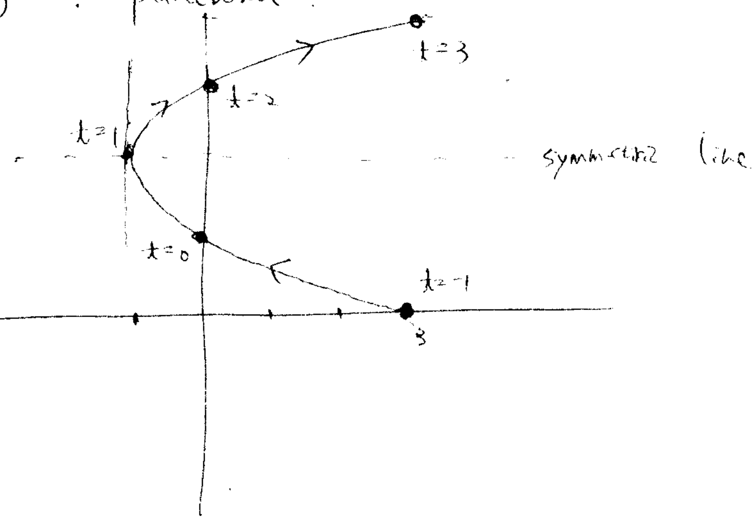
2. (a) Sketch the curve of the parametric equations $x = t^2 - 2t$, $y = t + 1$ for $-1 \le t \le 3$ and indicate with an arrow the direction in which the curve is traced as the parameter t increases.

$t = y - 1 \quad x = (y-1)^2 - 2(y-1) \quad$: parabola

$-1 \le t \le 3$. $\begin{cases} x(-1) = 1 + 2 = 3 \\ y(-1) = 0 \end{cases}$

$\begin{cases} x(0) = 0 \\ y(0) = 1 \end{cases} \qquad \begin{cases} x(1) = -1 \\ y(1) = 2 \end{cases}$

$\begin{cases} x(2) = 0 \\ y(2) = 3 \end{cases} \qquad \begin{cases} x(3) = 3 \\ x(3) = 4 \end{cases}$



(b) Find the points on the curve given in part (a) where the tangent line is horizontal or vertical- and draw these tangents on your plot of the curve.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2t-2}$$

$\frac{dy}{dx} = 0 \Rightarrow t \Rightarrow \infty$ no point for horizontal tangent

$\frac{dy}{dx} = \infty \Rightarrow 2t - 2 = 0$. $\underline{t = 1}$ $(x, y) = \underline{(-1, 2)}$

(c) Calculate the length of the curve given in part (a).

arc length $= \int_{-1}^{3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-1}^{3} \sqrt{(2t-2)^2 + 1}\, dt$

$u = 2t - 2$. $du = 2dt$

$$= \int_{-6}^{4} \sqrt{u^2+1} \cdot \frac{du}{2} = \frac{1}{2} \cdot \left[\frac{u}{2} \cdot \sqrt{1+u^2} + \frac{1}{2} \ln\left(u + \sqrt{1+u^2}\right) \right]_{-6}^{4}$$

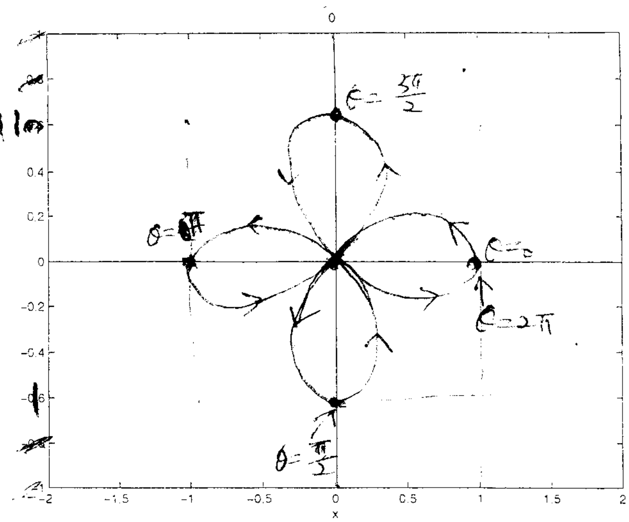
$$= \frac{1}{2}\left[\left(\frac{4}{2}\sqrt{17} + \frac{1}{2}\ln(4+\sqrt{17}) \right) - \left(\frac{-6}{2}\sqrt{37} + \frac{1}{2}\ln(-6+\sqrt{37}) \right) \right]$$

$$= \frac{1}{2}\left[2\sqrt{17} + 3\sqrt{37} + \frac{1}{2}\ln(4+\sqrt{17}) - \frac{1}{2}\ln(-6+\sqrt{37}) \right]$$ or use calc

From integral table $\int \sqrt{1+x^2}\, dx = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right) + C$ (trig subs $x = \tan t$)

3. (a) On the axes provided below, sketch the polar curve given by $r = \cos 2\theta$.

$0 \le \theta \le 2\pi$.



$r = \cos 2\theta = 0$. $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

$r(0) = \cos 0 = 1$

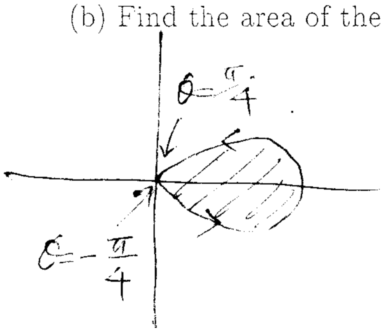
$r(\frac{\pi}{8}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$r(\frac{\pi}{4}) = 0$, $r(\frac{3\pi}{8}) = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

$\cancel{r(\frac{3\pi}{4})}$. $r(\frac{\pi}{2}) = -1$, $r(\frac{3\pi}{4}) = 0$

. . .

(b) Find the area of the region enclosed by one loop of the curve given in part (a).



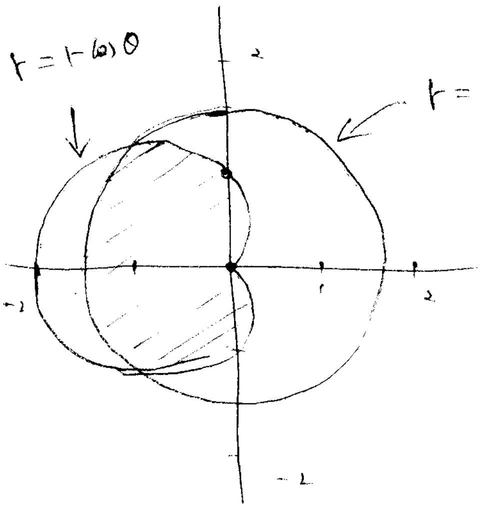
$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta = 2 \cdot \frac{1}{2} \int_0^{\pi/4} \cos^2 2\theta \, d\theta$$

$$= \int_0^{\pi/4} \frac{1 + \cos 4\theta}{2} \, d\theta = \left[\frac{\theta}{2} + \frac{\sin 4\theta}{8} \right]_0^{\pi/4}$$

$$= \underline{\underline{\frac{\pi}{8}}}$$

4. (a) Sketch the region that lies inside the curve $r = 1 - \cos(\theta)$ and outside the curve $r = 3/2$.



$r = 1 - \cos\theta$

$r(0) = 1 - 1 = 0$

$r(\frac{\pi}{4}) = 1 - \frac{\sqrt{2}}{2}$

$r(\frac{\pi}{2}) = 1$, $r(\frac{3\pi}{4}) = 1 + \frac{\sqrt{2}}{2}$

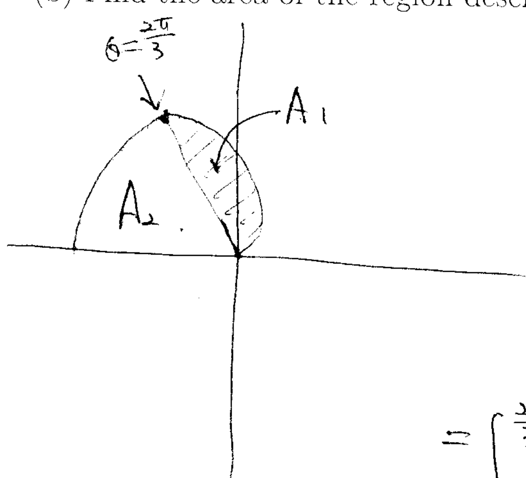
$r(\pi) = 2$, ...

intersection.

$1 - \cos\theta = 3/2$

$\cos\theta = -\frac{1}{2}$, $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.

(b) Find the area of the region described in part (a).



$A = 2(A_1 + A_2)$.

$$A_1 = \int_0^{\frac{2\pi}{3}} \frac{1}{2} r^2 \, d\theta = \int_0^{\frac{2\pi}{3}} \frac{1}{2} (1 - \cos\theta)^2 \, d\theta$$

$$= \int_0^{\frac{2\pi}{3}} \frac{1}{2}(1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \int_0^{\frac{2\pi}{3}} \frac{1}{2} - \cos\theta + \frac{1 + \cos 2\theta}{4} \, d\theta$$

$$= \left[\frac{\theta}{2} - \sin\theta + \frac{\theta}{4} + \frac{\sin 2\theta}{8} \right]_0^{\frac{2\pi}{3}} = \frac{\pi}{2} - \frac{9}{16}\sqrt{3}$$

$$A_2 = \frac{1}{6} \times \pi \times \left(\frac{3}{2}\right)^2 = \frac{3\pi}{3}$$

$$A = 2 \times (A_1 + A_2) = 2 \times \left(\frac{7\pi}{8} - \frac{9\sqrt{3}}{16} \right) = \frac{7\pi}{4} - \frac{9\sqrt{3}}{8}$$