Title: Random Media in Inverse Problems, Theoretical Aspects

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Inverse problem (IP) theory consists of making unknown media known. Random media (RM) theory is a method to model unknown media. Thus, ideally, IP and RM should not overlap: if the former is successful, then the latter is not necessary. Since most practical IPs are not ideal, however, mixing these two notions can sometimes be useful.

There are many ways that an IP is not ideal. An unfortunate feature shared by most of them is that they are ill-posed. The consequence is that small noise levels propagate to large errors in reconstructions unless a priori information is included in the reconstruction, i.e., unless an intractable problem is replaced by a simpler one. What this means in practice is that the unknown medium is made only partially known even when the best possible algorithm is being used. A second unfortunate feature of many nonlinear inverse problems is that the part that remains unknown influences the available measurements, and therefore inevitably the reconstruction of the part we want to claim as known.

It is there that randomness plays a role. There is no risk at modeling the unknown part as random when no better description is available. We can then assess the influence of that unknown part on the reconstruction of the part we want to claim as known. This allows for a framework of uncertainty quantification for the varied applications of IP.

At this level of abstraction, relatively little may be said quantitatively. Modeling the unknown part as random means figuring out a probability measure that best describes it. How this probability measure should be parameterized and the parameters chosen remains elusive. We consider here two examples in which a specific structure of randomness allows us to be fairly explicit about the probability measure.

High frequency noise and low frequency reconstructions

Consider for concreteness an elliptic operator $Lu = -u\text{''} + qu$ with an unknown potential $q = q(x)$ and measurements corresponding to the spectrum $\{\lambda_n\}$ of L augmented with, say, Dirichlet conditions on a bounded segment. Then λ_n grows like n^2. Let us assume that only the first N eigenvalues may be measured adequately. The most oscillatory corresponding eigenvector thus oscillates at a frequency of order N. Inverse Sturm-Liouville theory allows us to deduce that an order of $O(N)$ Fourier coefficients of q can be reconstructed satisfactorily. Unless very strong prior information on q is introduced, higher frequency components of q cannot be reconstructed.

Such components have an influence on the measured eigenvalues nonetheless. Because they are high frequency, we may approximate their influence by looking at their limiting behavior when $N \to \infty$. This in turn allows us to infer the influence of these non-recoverable components on the reconstruction of the low-frequency components. Minimum variance reconstructions may then be devised, whose role is to limit as much as possible the influence of the unknown, non-recoverable, components. This serves as an example of application of the theory of differential equations with random coefficients to improve the solution of an inverse problem. We refer the reader to [8] and references there for details.

IP with RM or the search for Stable Observables

Our second example is motivated by the reconstruction of inclusions buried in heterogeneous media (HM). Applications include biomedical imaging, seismic exploration in geophysics and non-destructive testing of materials. We assume the medium probed by (acoustic, electromagnetic, or elastic) waves and measurements consisting of scattered waves. We think of a situation where HM is of little interest to us. Only the imaging of the inclusion matters. In the unlikely event that HM is known, then the invariance of the wave equation by time reversal provides the right solution to the inverse problem: back-propagate available data solving the wave equation on a computer and they will reconstruct the inclusion [10].

When HM is not known, simply ignoring it may provide very inaccurate reconstructions. We then have two paths forward. We can either reconstruct HM explicitly or we need to find a *new* inverse problem in which the influence of HM is minimal. Unless very accurate wave measurements are available, the first option is often not available. It then makes sense to model HM as RM. This is the scenario we consider for the rest of the chapter.

The main difficulty we now face is that wave measurements strongly depend on the realization of RM. As a consequence, reconstructions are very much affected by the specific details of HM [9] and are therefore *statistically unstable*. The original inverse wave problem thus needs to be replaced by a *statistically stable* one, i.e., one where the reconstruction of the buried inclusion will depend as little on the realization of RM as possible. Stable reconstructions require stable functionals of the available wave measurements. By analogy with quantum mechanics, we will refer to such functionals as *observables*. The ideal inverse problem, when it exists, then becomes: how does one reconstruct the buried inclusion from knowledge of these statistically stable observables?

Field-field Correlations are Stable Observables

A very fruitful approach in the search for stable observables is to consider the broad family of field-field correlations. Here field refers to the solution of the wave equation. The fields themselves are not statistically stable [9]. In several interesting settings, it has been shown that correlations could indeed play the role of stable observables. Moreover, such correlations often solve closed-form, kinetic, equations in which the buried inclusion acts as a constitutive parameter. The "new" inverse problem thus becomes an inverse kinetic problem, which in some cases enjoys reasonably favorable reconstruction properties [2].

For concreteness, with d spatial dimension, p pressure and $\mathbf{v}$ velocity, let the $(d+1)-$vector $\mathbf{u} = (p, \mathbf{v})$ solve the following system of acoustic wave equations

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0, \qquad \kappa(\mathbf{x})\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{v} = 0, \qquad \mathbf{x}\in\mathbb{R}^d,\ t>0, \tag{1}$$

supplemented with initial conditions $p(t=0,\mathbf{x})$ and $\mathbf{v}(t=0,\mathbf{x})$. Here ρ is density and $\kappa(\mathbf{x})$ a highly heterogeneous compressibility. The buried inclusion may be modeled as a variation in $\kappa(\mathbf{x})$ as well. We probe the system with high frequency waves, i.e., with wavelength $\lambda = \varepsilon L \ll L$, where L is the overall size of the domain of interest. This is modeled by

$$p(t=0,\mathbf{x}) = p_0\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right), \qquad \mathbf{v}(t=0,\mathbf{x}) = \mathbf{v}_0\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right).$$

Whereas fields $\mathbf{u}$ are quite sensitive to the heterogeneities in $\kappa(\mathbf{x})$, there are several situations in which quadratic quantities in the field are stable observables. Because fields oscillate at the scale ε, correlations need to occur at this scale as well.

Let $\mathbf{u}^\phi(t,\mathbf{x})$ for $\phi = 1,2$ be solutions for possibly different initial conditions and possibly different RM modeled by $\kappa^\phi(\mathbf{x})$. The Fourier transform of the correlation function with respect to the offset variable is called the matrix-valued *Wigner transform* and is defined for $1 \leq \psi, \phi \leq 2$ by

$$W_\varepsilon^{\psi\phi}(t,\mathbf{x},\mathbf{k}) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\,\mathbf{k}\cdot\mathbf{y}}\,\mathbf{u}^\psi(t,\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2})\otimes\mathbf{u}^\phi(t,\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2})\,d\mathbf{y}. \qquad (2)$$

Kinetic Models and Statistical Stability

When $\phi = \psi = 1$, then we observe that the trace of the integral of W_ε over wavenumbers $\mathbf{k}$ provides $|\mathbf{u}|^2(t,\mathbf{x})$, a quantity comparable to the wave energy density. In this case, $W_\varepsilon^{\psi\phi}(t,\mathbf{x},\mathbf{k})$ should be interpreted as a phase-space resolution of the wave energy density. In the limit $\varepsilon \to 0$, high frequency waves behaves as particles and we thus expect W_ε to approximately solve a kinetic equation. In dimension $d = 1$, this picture is incorrect as wave tend to localize rather than transport according to a kinetic model; see [11]. In dimension $d \geq 2$, this picture is more or less correct for appropriate RM.

The limiting, deterministic, kinetic equation that W_ε in (2) satisfies in the limit $\varepsilon \to 0$ strongly depends on the structure of $\kappa(\mathbf{x})$, which we assume of the form

$$\kappa(\mathbf{x}) = \kappa_0(\mathbf{x}) + \sigma_0\kappa_1\left(\frac{\mathbf{x}}{l_c}\right), \quad \mathbb{E}\{\kappa(\mathbf{x})\} = \kappa_0(\mathbf{x}), \quad \mathbb{E}\{\kappa_1(\mathbf{x})\kappa_1(\mathbf{y})\} = R(\mathbf{x}-\mathbf{y}), \quad (3)$$

where $\mathbb{E}$ is mathematical expectation and R the correlation function of κ_1. Different regimes arise depending on the relative size of the (adimensionalized) correlation length l_c with ε. For instance, in the regime $l_c = \sigma_0^2 = \varepsilon \ll 1$, the kinetic equation is a radiative transfer equation [13]. In the regime $\varepsilon \ll l_c = \sigma_0^2 \ll 1$, the kinetic equation is a Fokker-Plank equation. We refer the reader to the recent review [4] for details on the derivations of the limiting kinetic models and their levels of mathematical rigorousness.

Our observable W_ε is a random object that we approximate by W the solution to a deterministic kinetic equation. In which sense does then W_ε converge to W? When $\kappa = \kappa_0$ is not random, it is known that the convergence of W_ε to its limit can only occur in a weak sense [12], i.e., that $\langle W_\varepsilon, \varphi\rangle$ converges to $\langle W, \varphi\rangle$ for a sufficiently smooth test function φ. It turns out that for the random models considered above, the random object $\langle W_\varepsilon, \varphi\rangle$ converges in *probability* to the *deterministic* object $\langle W, \varphi\rangle$. This means that:

$$\mathbb{P}\Big(|\langle W_\varepsilon(t), \varphi\rangle - \langle \mathbb{E}\{W_\varepsilon(t)\}, \varphi\rangle| \geq \delta\Big) \to 0, \qquad \text{uniformly in } t \text{ on compact intervals.}$$

This is precisely what we were looking for to reconstruct our inclusion. We have devised an observable W_ε that in the limit $\varepsilon \to 0$ solves a kinetic equation where $\kappa_0(\mathbf{x})$ is a constitutive parameter. The reconstruction of the inclusion then becomes an inverse kinetic problem.

Scintillation function and accuracy of the reconstruction

How well can one expect to reconstruct the inclusion? The resolution depends on the structure of the kinetic inverse problem itself [2] but also on the amount of noise in the “kinetic” data, i.e., on the discrepancy between W_ε and its limit W. A natural gauge for the statistical instability of W_ε is the so-called scintillation function defined as

$$J_\varepsilon(t, \mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{p}) = \mathbb{E}\{W_\varepsilon(t, \mathbf{x}, \mathbf{k}) W_\varepsilon(t, \mathbf{y}, \mathbf{p})\} - \mathbb{E}\{W_\varepsilon(t, \mathbf{x}, \mathbf{k})\}\mathbb{E}\{W_\varepsilon(t, \mathbf{y}, \mathbf{p})\}, \qquad (4)$$

i.e., the statistical correlation function of the Wigner transform (assumed to be scalar to simplify notation).

There are relatively few results on the behavior of J_ε as $\varepsilon \to 0$. In simplified regime of wave propagation, the scintillation function is well understood; see [4, 6]. Unfortunately, its behavior is rather complicated and strongly depends on the phase-space structure of the initial conditions $\mathbf{u}(t = 0)$, on the size of the detector array as well as on the correlation function of the random medium.

Which observables should we choose?

Let us return to the reconstruction of the inclusion. We have obtained different kinetic models for different types of correlations. It turns out that in some situations, some correlations are more sensitive than others to the presence of the inclusion. Such correlations should be used to maximize signal to noise ratios (SNR).

Consider the example of wave fields measured in the absence $\mathbf{u}^1$ and in the presence $\mathbf{u}^2$ of the inclusion and let $W_\varepsilon^{\phi,\psi}(t, \mathbf{x}, \mathbf{k})$ be the cross-correlation. We have seen that $W_\varepsilon^{\phi,\psi}(t, \mathbf{x}, \mathbf{k})$ was a stable observable for all values of $1 \le \phi, \psi \le 2$.

The simplest inversion procedure should then be based on using the model for $W^{2,2}(t, \mathbf{x}, \mathbf{k})$. More plausibly, only $\int_{\mathbb{R}^d} W^{2,2}(t, \mathbf{x}, \mathbf{k}) d\mathbf{k} = \mathbf{u}^2(t, \mathbf{x}) \otimes \mathbf{u}^2(t, \mathbf{x})$ may be measured in practice. This method is the least expensive experimentally as it only requires energy measurements in the presence of the inclusion. Its applicability is however limited by following requirement: the influence of the inclusion has to be larger on the data than the statistical instability $W_\varepsilon^{2,2} - W^{2,2}$. It is therefore prone to high SNR levels.

A remedy to these high SNR is to use the *differential* measurement $\delta W_\varepsilon := W_\varepsilon^{2,2} - W_\varepsilon^{1,1}$ provided that they are available as they require probing the medium in the presence *and* in the absence of the inclusion. Such measurements have significantly lower SNR as, heuristically, the random influence of signals that do not visit the inclusion cancels out in δW_ε.

A third possibility is to use the cross-correlation $W_\varepsilon^{1,2}$, which is technologically the most difficult measurement as it necessitates to measure the two vector fields $\mathbf{u}^\phi$ for $\phi = 1, 2$ and then cross-correlate them. In highly disordered media, i.e., when the transport mean free path is small compared to L, then such observables display the largest SNR. Indeed, an inclusion of radius R in such an environment will have an influence on the energy difference of order $W^{1,1} - W^{2,2} = O(R^d)$ while its influence on the cross-correlation will be of order $W^{1,1} - W^{1,2} = O(R^{d-2})$ in dimension $d \geq 3$ [5].

For experimental and numerical validations of radiative transfer equations and their applications to the reconstruction of buried inclusions, we refer the reader to [1, 3, 5, 7] and their references.

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