Time reversal by time-dependent perturbations

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Abstract

We consider the time-reversal of waves in time-dependent media. The constitutive parameters of a wave equation are assumed constant in time during intervals $(0, T-\tau) \cup (T+\tau, 2T)$ for $0 < \tau \ll T$ and time-varying during the time interval $(T-\tau, T+\tau)$ to model a quasi-instantaneous time mirror. We show that under appropriate hypotheses, a time-reversed signal is generated by such time-dependent fluctuations. At time $2T$, the time-reversed signal is an appropriate differentiation of the original initial condition when the time-varying fluctuations are independent of space. In the case of spatially varying fluctuations, we show that the time-reversed signal enjoys stronger refocusing properties when propagation occurs in a highly heterogeneous environment, as in the case of *classical* time-reversal.

This paper offers additional theoretical justifications to the experimental results obtained in [1] and proposes potential extensions. We present numerical simulations that corroborate and quantify the theoretical predictions.

1 Introduction

This paper concerns time reversal phenomena of waves propagating in time-dependent media. It is known that wave equations with time-independent coefficients are invariant by time reversal in the sense that the wave propagation operator from time 0 to time T is a unitary operator $\mathcal{U}(T)$ in an appropriate metric realizing energy conservation, and that wave propagation from time T to time 0 is described by $\mathcal{U}^*(T) = \mathcal{U}(-T)$, which involves wave propagation in the *same* medium albeit with time-reversed initial conditions. As a consequence, an initial fluctuation propagated by $\mathcal{U}(T)$, recorded at time T, *time-reversed* and then propagating in the *same* medium for a duration T, and *time-reversed* one more time, will be exactly equal to the original fluctuation. Said differently, if $\mathcal{T}$ is the time-reversion operation (with $\mathcal{T}^2 = I$), then $\mathcal{U}^*(T) = \mathcal{U}(-T) = \mathcal{T}\mathcal{U}(T)\mathcal{T}$ so that $\mathcal{T}\mathcal{U}(T)\mathcal{T}\mathcal{U}(T) = I$ the identity operator.

A notable corollary to time-reversal is the refocusing properties of partially recorded waves. Let $\chi(x)$ describe a spatial domain (a detector array) where a signal is recorded. The operator modeling the time-reversed signal is $\mathcal{T}\mathcal{U}(T)\mathcal{T}\chi(x)\mathcal{U}(T)$. This is no longer identity. A striking feature of *classical* time-reversal is that the latter operator 'resembles' identity much more closely when propagation occurs in a heterogeneous medium than when it does in a homogeneous one. Heuristically, heterogeneities create multiple reflections (multi-pathing), and the

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recorded signal by $\chi(x)$ then involves a superposition of many more paths that will interfere coherently during the backpropagation stage modeled by $\mathcal{T}\mathcal{U}(T)\mathcal{T}$. We refer the reader to [4, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17] and their multiple references for descriptions and background on time reversal in heterogenous environments.

The main objective of this paper is to analyze time reversal phenomena where the measurement/time reversal operator $\mathcal{T}\chi(x)$ is 'replaced' by a short-duration large-amplitude *time-dependent fluctuation* ('instantaneous' time mirror [1]) in the properties of the underlying medium. For a convincing experimental realization of the phenomenon when $\chi(x) \equiv 1$, we refer the reader to the recent paper [1].

Let us consider time-dependent fluctuations on the support of $\chi(x)$ in the τ-'vicinity' of time T and call their effect on wave propagation as the operator $\mathcal{F}_{T,\chi}$. We then wish to analyze the wave field $\mathcal{T}\mathcal{U}(t)\mathcal{F}_{T,\chi}\mathcal{U}(T)$. Under appropriate hypotheses on the time-dependent fluctuations, we have $\mathcal{F}_{T,\chi} = I + \alpha(\delta\mathcal{F})_{T,\chi}$, where α models the fluctuations' strength, and we will show that the latter operator decomposes into four components. The first one is simply the unperturbed solution modeled by $\mathcal{T}\mathcal{U}(2T)$. The second contribution is another forward-propagating component proportional to α. The third contribution is the main interest of this paper, and may heuristically be well-approximated (when τ is sufficiently small as we will see) by $\alpha\mathcal{T}\mathcal{U}(T)\mathcal{D}\mathcal{T}\mathcal{U}(T) = \alpha\mathcal{D}\mathcal{T}\mathcal{U}(T)\mathcal{T}\mathcal{U}(T) = \alpha\mathcal{D}$, where $\mathcal{D}$ is a differentiation operator described in detail in the text. In other words, we obtain a back-propagating signal, which up to a scaling α and a (Fourier domain scaling) local differentiation, is the original initial condition. The fourth component is a remainder, which we show is of order $\alpha\tau$ (in an idealized setting).

In other words, if the solution is considered at time $2T$ on the support[1] of the initial condition, then what we observe is a local operator $\alpha\mathcal{D}$ applied to the initial condition up to negligible contributions. This is entirely consistent with the experimental observations in [1] of wave refocusing in the presence of instantaneous time mirrors. Note that the wave field energy is not conserved (and here increases) in such a setting [1] or in general in the presence of time-dependent variations of the underlying medium [18].

We model wave propagation by a system of acoustic equations. For appropriate short-time fluctuations, we justify the above decomposition, first for spatially independent time-dependent fluctuations ($\chi \equiv 1$, which corresponds to the setting in [1]), and second for spatially varying time-dependent fluctuations (with $\chi(x)$ a compactly supported function). In the latter case, we also consider wave propagation in spatially heterogeneous media in order to observe the *classical* enhancement of the refocusing of time reversed waves in rich multi-path environments. These results may be the most promising from a practical point of view. On the one hand, *classical* time-reversal requires measuring, time-reversing, and back-propagating wave fields, which are measurement- and memory-intensive procedures [14, 15]. On the other hand, full domain time-dependent fluctuations as in [1] (corresponding to $\chi(x) \equiv 1$) may not be feasible. Our results show that even if the coefficients fluctuate in time only on a small spatial zone (with $\chi(x)$ of small support), and provided that propagation occurs in a sufficiently heterogeneous environments, then the time-reversed signal $\alpha\mathcal{D}f(x)$ captures the main features of the initial signal $f(x)$ without the more complex apparatus of *classical* time-reversal.

The paper is structured as follows: in section 2, we introduce our model of wave propagation and derive a decomposition of the wavefield that helps us understand the properties of the propagating and refocusing fields. Section 3 focuses on the case of spatially independent time-dependent perturbations. Several numerical simulations illustrate the theoretical

[1] Consider a localized support on a domain of diameter d with $d \ll cT$, where c is speed propagation and T the central time of the localized time-dependent fluctuations.

results. Section 4 generalizes the analysis to the setting of spatially localized time-dependent perturbations. We characterize the refocusing signal both in the settings of homogeneous and heterogenous underlying media of propagation. All proofs are postponed to section 5. The paper is concluded by an appendix where we consider several interesting cases of time-dependent fluctuations.

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2 Acoustic system and time-dependent media

Our starting point is the acoustic wave equation, written as a first-order hyperbolic system

$$\rho(t,x)\frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0, \qquad \kappa(x)\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{v} = 0, \qquad x \in \Omega, \tag{1}$$

where the unknowns are the velocity field $\mathbf{v}(t,x)$ and the acoustic pressure $p(t,x)$, and the known constitutive coefficients are the density ρ and the compressibility κ. The system (1) is equipped with (smooth) initial conditions $\mathbf{v}(0,x) = \mathbf{v}_0(x)$ and $p(0,x) = p_0(x)$ at $t = 0$. The domain $\Omega \subset \mathbb{R}^3$ can be bounded or not. Appropriate boundary conditions on $\partial\Omega$ are added, say Dirichlet or Neumann boundary conditions, if necessary. The analysis of sections 2 and 3 is presented in three spatial dimensions for concreteness of notation. It holds in any dimension $d \geq 1$. The results of section 4, focusing in heterogeneous media, hold only in dimension $d \geq 2$.

We are interested in analyzing the effects of short and large variations (approximated by a δ function in time) in the coefficients (ρ, κ) on wave propagation. Such terms are compatible with the wave equation in only specific cases as we shall see. In this paper, we mainly assume that density varies while compressibility remains time-independent, and more precisely introduce the model:

$$\rho^{-1}(t,x) = \rho_0^{-1}(x)\big(1 + V(t,x)\big), \tag{2}$$

where ρ_0 is the background (i.e. unperturbed) density, while $V(t,x)$ is a time-dependent perturbation that may be spatially localized. More general models are considered in the appendix. V is chosen so that ρ remains strictly positive while ρ_0 and κ are smooth functions. For $T > 0$ and $0 < \tau \leq T$ given, and η a positive function with integral one, we define

$$\eta_\tau(t) = \frac{1}{\tau}\eta\left(\frac{t}{\tau}\right), \qquad supp\,\eta \subset \left[-\frac{1}{2}, \frac{1}{2}\right].$$

The function η_τ is an approximation of a delta function at $t = 0$. We consider spatially independent perturbations $V(t,x) = V(t) = \alpha\eta_\tau(t - T)$ in section 3, and spatially dependent ones of the form

$$V(t,x) = \alpha\eta_\tau(t-T)\chi(x), \tag{3}$$

in section 4. The parameter $\alpha > 0$ models the strength of the perturbation, and χ is a function on Ω representing the spatial location of the perturbation; see the end of this section and the appendix for a discussion on more general choices of time-dependent coefficients. The theory we present is of interest in the regime where $\tau \ll T$ and $\alpha\tau \ll T$.

Note that the system (1) can be recast as the scalar second-order wave equation

$$\frac{\partial^2 p}{\partial t^2} = \kappa(x)^{-1}\nabla\cdot\big(\rho(t,x)^{-1}\nabla p\big), \tag{4}$$

with initial conditions $(p(t=0), \partial_t p(t=0)) = (p_0, p_1)$, where $p_1 = -\kappa^{-1}\nabla\cdot\mathbf{v}_0$. It is more convenient for us to work directly with the system (1) to obtain the following expression.

Decomposition of the wavefield. We first recast the system (1) as

$$A(x)\frac{\partial \mathbf{u}}{\partial t} + D^j \frac{\partial \mathbf{u}}{\partial x_j} = V(t,x)\mathbf{S}, \tag{5}$$

where $V(t,x)$ is defined in (3) and $\mathbf{u} = (\mathbf{v}, p)$, $\mathbf{S} = -(\nabla p, 0)$ (which are both considered as column vectors), $A = \mathrm{Diag}(\rho_0, \rho_0, \rho_0, \kappa)$, and $(D^j)_{mn} = \delta_{m4}\delta_{nj} + \delta_{n4}\delta_{mj}$, with $j = 1, 2, 3$, and $m, n = 1, \cdots, 4$. Here and below, we use the summation convention over repeated indices. Equation (5) is equipped with the initial condition $\mathbf{u}_0 = (\mathbf{v}_0, p_0)$. It is convenient to describe wave propagation in terms of the Green's function of (5) and its adjoint, defined respectively by

$$A(x)\frac{\partial \mathbf{G}(t,x,y)}{\partial t} + D^j \frac{\partial \mathbf{G}(t,x,y)}{\partial x_j} = 0, \qquad \mathbf{G}(0,x,y) = I\delta(x-y), \tag{6}$$

where I is the 4×4 identity matrix, and

$$\frac{\partial \mathbf{G}^*(t,x,y)}{\partial t} A(x) + \frac{\partial \mathbf{G}^*(t,x,y)}{\partial x_j} D^j = 0, \qquad \mathbf{G}^*(0,x,y) = A^{-1}(x)\delta(x-y).$$

The adjoint is used in the proof of the main result of section 4. We will use the notation

$$\mathbf{G}_t(\mathbf{u})(x) = \int_\Omega \mathbf{G}(t,x,y)\mathbf{u}(y)dy. \tag{7}$$

The unperturbed solution to (5), i.e. with $V = 0$, is denoted by $\mathbf{U} = (\mathbf{V}, P)$. The main tool in our analysis is the following decomposition of the wavefield $\mathbf{u}$ into various propagating and refocusing contributions.

Lemma 2.1 *Let* $\boldsymbol{\phi}_0 = -(\rho_0^{-1}\nabla p_0, \kappa^{-1}\nabla \cdot \mathbf{v}_0) = \partial_t \mathbf{U}(t=0)$ *and* $\Gamma = \mathit{Diag}(-1,-1,-1,1)$. *Then, the solution* $\mathbf{u}$ *admits the decomposition*

$$\mathbf{u}(t) = \mathbf{U}(t) + \mathbf{u}_R(t) + \mathbf{u}_F(t) + \mathbf{R}^\tau(t), \tag{8}$$

where

$$\begin{aligned}
\mathbf{u}_R(t,x) &= -\frac{\alpha}{2}\int_0^t \int_\Omega \eta_\tau(s-T)\boldsymbol{\mathcal{K}}(t,s,x,z)\boldsymbol{\phi}_0(z)dsdz \\
\boldsymbol{\mathcal{K}}(t,s,x,z) &= \int_\Omega \mathbf{G}_{t-s}(x,y)\Gamma \mathbf{G}_s(y,z)\chi(y)dy,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{u}_F(t) &= \frac{\alpha}{2}\int_0^t \mathbf{G}_{t-s}(V_s \partial_t \mathbf{U}_s)ds \\
\mathbf{R}^\tau(t) &= \int_0^t \mathbf{G}_{t-s}(V_s A^{-1}(\mathbf{S}_s - \mathbf{S}_s^0))ds \qquad \textit{with} \qquad \mathbf{S}^0(t) = -(\nabla P(t), 0).
\end{aligned}$$

The (very classical) proof of the decomposition is postponed to section 5. In the lemma above, the matrix Γ describes the time reversal operation, that changes the sign of the velocity $\mathbf{v}$ while keeping the pressure p fixed.

The first term in (8), $\mathbf{U}$, is the unperturbed solution corresponding to $V \equiv 0$. The next two terms are the leading contributions generated by the perturbation (3) when $\alpha \neq 0$. The contribution $\mathbf{u}_R$ is the (time-reversed) refocusing signal and the main object of interest in this paper. When evaluated at time $t = 2T$, where T is the centered time of the time-dependent perturbation, we expect $\mathbf{u}_R$ to be directly related to the initial conditions $\mathbf{U}_0$ and more precisely to $\partial_t \mathbf{U}(t=0)$ when $\tau \ll T$. The contribution $\mathbf{u}_F$ is the forward-propagating signal created by the perturbation, which as seen in the lemma is proportional to α. Finally, the remainder $\mathbf{R}^\tau$ captures the temporal 'width' of the time-dependent fluctuation and will be shown to be of order $\alpha\tau$ in a simplified setting.

Remarks on the choice of time-dependent coefficients. Before analyzing the above terms when $\chi \equiv 1$ in the next section and χ with compact support in the following section, we make a few comments on the choice of temporal perturbations.

We first remark that we perturb ρ^{-1} in (2) and not ρ. When the fluctuations are smooth in time, there is no significant difference between these choices: solutions of wave equations with smooth spatio-temporal (positive) coefficients are well defined. However, as the form of $\mathbf{u}_R$ above and the results to follow show, interesting refocusing occurs when $\tau \ll T$. This means that the fluctuations need to be large over a short time span to have an interesting effect. Whether we are capable to find a macroscopic theory for such an effect is essentially what dictates our choices of coefficients. This paper focuses on the choice described in (2) and considers more general choices in the appendix.

Suppose for simplicity that $V(t,x) = \delta(t-T)$ with δ the Dirac delta function. Perturbing ρ^{-1} leads to the term $\delta(t-T)\nabla p$ in the first equation in (1), which makes sense mathematically only when ∇p is *continuous* in the time variable. Heuristically, the latter is true by the following arguments (we prove this fact later in a simple setting): since $\partial_t \mathbf{v}$ is proportional to a Dirac function according to (1), then its integral $\mathbf{v}$ is a discontinuous function in time. The second equation in (1) shows in turn that p is continuous in time, and it can be assumed that the continuity extends to ∇p when the initial conditions are smooth. It follows that the term $\delta(t-T)\nabla p$ makes sense from a mathematical standpoint.

Perturbing ρ would lead to the term $\delta(t-T)\partial_t \mathbf{v}$, which cannot be justified mathematically as it does not seem possible to conclude from the wave equation that $\partial_t \mathbf{v}$ is continuous and this is probably incorrect. Hence the perturbation of ρ^{-1}.

Continuity in time of p is also crucial for the quality of the refocusing: in (8), the terms $\mathbf{u}_R$, $\mathbf{u}_F$ and $\mathbf{R}^\tau$ are a priori of the same amplitude (proportional to α). It is the fact that p is smooth in the t variable that allows us to show that $\mathbf{R}^\tau$ is indeed small compared to the signal of interest $\mathbf{u}_R$. See the appendix for additional models of temporal fluctuations.

Although we do not pursue this in detail here, we could similarly consider a system with a time-dependent perturbation in the compressibility and not in the density, i.e.

$$\rho(x)\frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0, \qquad \kappa(t,x)\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = 0, \qquad x \in \Omega.$$

Supposing that the initial velocity field $\mathbf{v}_0$ is irrotational, the first equation above shows that $\mathbf{v}(t)$ remains irrotational at all times, and can then be expressed as $\mathbf{v}(t) = \nabla\phi(t)$, where ϕ is the velocity potential. Then, the quantity $w = \Delta\phi$ satisfies

$$\frac{\partial^2 w}{\partial t^2} = \nabla \cdot \big(\rho(x)^{-1}\nabla(\kappa(t,x)^{-1}w)\big).$$

With $\kappa^{-1}(t,x) = \kappa_0^{-1}(x)(1+V(t,x))$, we obtain a similar theory to the one described in the paper since it can be shown that w is continuous at time T where the jump occurs.

For the same reasons as explained above, we are not able to singularly perturb both ρ^{-1} and κ^{-1} and give a meaning to (5). Indeed, as we show in the appendix, it is in general false that *both* ∇p and $\nabla \cdot \mathbf{v}$ are continuous functions in time, and as a consequence the definition of (at least one of) $\delta(t-T)\nabla p$ and $\delta(t-T)\nabla \cdot \mathbf{v}$ is unclear.

When the fluctuations are spatially localized, for instance with $\chi(x)$ of compact support, the formalism described in (5), with fluctuations approximated by a delta-function, is an essential ingredient in our analysis in section 4. It is nevertheless interesting to understand the structure of the time-reversed signal when *both* coefficients ρ^{-1} and κ^{-1} are perturbed. This is done in some detail in the appendix in the setting of global time dependent perturbations (with $\chi(x) \equiv 1$). It can then be shown that time-dependent fluctuations may couple propagating modes only with their 'time-reversed' versions. In particular, we show that fluctuations such that $\rho(t)/\kappa(t) = \zeta^2$ is constant (i.e., with constant impedance) generate no time-reversed signals, in agreement with physical intuition. We refer the reader to the appendix for the details.

3 Spatially-independent perturbation

The perturbation V in (3) is assumed independent of position (with $\chi \equiv 1$) and thus has the form $V(t) = \alpha\eta_\tau(t)$, for η_τ an approximation of the delta function. We extend the Green's function $\mathbf{G}_t$ to negative values of t by solving (6) for $t < 0$, and have $\mathbf{G}_{-t}\Gamma = \Gamma\mathbf{G}_t$ with $\Gamma = \mathrm{Diag}(-1,-1,-1,1)$; in other words the time-reversal operator $\mathcal{T}$ of the introduction takes the form Γ for the acoustic system of equations. In the same way, we extend $\partial_t\mathbf{U}(t)$ to negative times by solving the wave equation for $t<0$ with initial condition $\boldsymbol{\phi}_0 = -(\rho_0^{-1}\nabla p_0, \kappa^{-1}\nabla\cdot\mathbf{v}_0) = \partial_t\mathbf{U}(t=0)$. It follows that, for all $t \in \mathbb{R}$,

$$\partial_t\mathbf{U}(t) = \mathbf{G}_t(\boldsymbol{\phi}_0). \tag{9}$$

The main result of the section is the following analysis of the terms introduced in the decomposition (8).

Theorem 3.1 *The terms $\mathbf{u}_R(2T)$ and $\mathbf{u}_F(t)$ for $t \geq T + \frac{\tau}{2}$ admit the expressions:*

$$\mathbf{u}_R(2T) = -\frac{\alpha}{2}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \eta_\tau(s)\Gamma\partial_t\mathbf{U}(2s)ds$$

and

$$\mathbf{u}_F(t) = \frac{\alpha}{2}\partial_t\mathbf{U}(t).$$

Moreover, under smoothness assumptions on the initial conditions and constant coefficients ρ_0 and κ_0 with $\Omega = \mathbb{R}^3$, then the remainder R^τ is of order τ (the temporal width of the fluctuations) as described in (10) *in Lemma 3.2 below.*

Since the proof of the expressions for $\mathbf{u}_R$ and $\mathbf{u}_F$ describes the time reversal symmetry of the wave equation that leads to the result, we present it here.

Proof. Starting from Lemma 2.1, we find, after the change of variables $s \to s+T$,

$$\mathbf{u}_R(2T,x) = -\frac{\alpha}{2}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\int_\Omega \eta_\tau(s)\boldsymbol{\mathcal{K}}(2T, s+T, x, z)\boldsymbol{\phi}_0(z)dsdz,$$

with, when $s \geq 0$,

$$\begin{aligned}
\boldsymbol{\mathcal{K}}(2T, s+T, x, z) &= \int_\Omega \mathbf{G}_{T-s}(x,y)\Gamma\mathbf{G}_{T+s}(y,z)dy \\
&= \int_\Omega\int_\Omega \mathbf{G}_{T-s}(x,y)\Gamma\mathbf{G}_{T-s}(y,z')\mathbf{G}_{2s}(z',z)dydz' \;=\; \Gamma\mathbf{G}_{2s}(x,z).
\end{aligned}$$

Above, we used the time reversibility of the acoustic wave equation which yields that

$$\int_\Omega \Gamma \mathbf{G}_{T-s}(x,y)\Gamma \mathbf{G}_{T-s}(y,z')dy = \delta(x-z')I.$$

When $s \leq 0$, with obtain with a similar calculation that

$$\boldsymbol{\mathcal{K}}(2T, s+T, x, z) = \mathbf{G}_{-2s}(x,z)\Gamma.$$

Using the fact that $\mathbf{G}_{-2s}\Gamma = \Gamma \mathbf{G}_{2s}$, $\forall s \in \mathbb{R}$, we arrive at

$$\mathbf{u}_R(2T,x) = -\frac{\alpha}{2}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\int_\Omega \eta_\tau(s)\Gamma \mathbf{G}_{2s}(x,z)\boldsymbol{\phi}_0(z)dsdz,$$

which, together with (9), yields the expression stated in the theorem. Regarding $\mathbf{u}_F(t)$ for $t \geq T+\tau/2$, we have, since $\partial_t \mathbf{U}$ solves (5) with $V=0$,

$$\mathbf{u}_F(t) = \frac{\alpha}{2}\int_0^t \eta_\tau(s-T)\mathbf{G}_{t-s}(\partial_t \mathbf{U}(s))ds = \frac{\alpha}{2}\int_0^t \eta_\tau(s-T)\partial_t \mathbf{U}(t)ds = \frac{\alpha}{2}\partial_t \mathbf{U}(t).$$

This ends the proof. $\square$

We now comment on the properties of the different terms that appear in the decomposition of $\mathbf{u}(2T)$ in (8). The first term $\mathbf{U}(2T)$ is nothing but the unperturbed solution. Suppose that the initial condition $(\mathbf{v}_0, p_0)$ has a bounded support and, for the moment, that the background is constant. When Ω is unbounded, or when T is such that reflections at the boundary $\partial\Omega$ do not reach the support of $(\mathbf{v}_0, p_0)$ at $t = 2T$, the contribution of $\mathbf{U}(2T)$ on this support vanishes. In the general case where the background is not constant, there could be some non-zero contributions of $\mathbf{U}(2T)$ to $\mathbf{u}(2T)$, but these are expected to be small compared to $\mathbf{u}_R$ provided the background is non-trapping around the support of the initial condition. The third term $\mathbf{u}_F$ is the forward solution created by the perturbation, and is proportional to $\partial_t \mathbf{U}(2T)$. Again, this contribution is expected to be either zero or quite small on the support of the initial condition.

The last term $\mathbf{R}^\tau$ is a remainder that is negligible when $\alpha\tau$ is small compared to a quantity measuring wavenumbers in the initial condition (see equation (10) below, which shows that if M is the largest wavenumber in the initial condition, $\mathbf{R}^\tau$ is small provided $C\alpha\tau M^6 \ll 1$, for C a universal constant). It is actually zero when the perturbation is a delta function: suppose that ∇p is continuous in time since it is a condition for the equations to make sense (this fact is proved below in a simple setting); then $\mathbf{R}^\tau$ is proportional to $\nabla p(T) - \nabla P(T)$, which we rewrite as $\nabla p(T) - \nabla p(T^-) + \nabla p(T^-) - \nabla P(T)$. The second part is zero since $\nabla p(T^-) = \nabla P(T^-)$ as we are right before the perturbation kicks in and ∇P is continuous, and the first part is zero as well since ∇p is continuous. For a general function η_τ, write the term $\mathbf{S} - \mathbf{S}^0$ in the definition of $\mathbf{R}^\tau$ as, for any $s \in [T - \frac{\tau}{2}, T + \frac{\tau}{2}]$,

$$\nabla(p(s) - P(s)) = \nabla(p(s) - p(s-\tau)) + \nabla(p(s-\tau) - P(s)).$$

As we are before the perturbation, the second term in the r.h.s. is $\nabla P(s-\tau) - \nabla P$. Since P is smooth, this second term is negligible provided τ is small compared to a parameter estimating ∇P. The same applies to the first term provided ∇p is continuous in time.

When ρ_0 and κ are constant, and $\Omega = \mathbb{R}^3$, we are able to make these arguments precise in the following lemma ($\widehat{f}$ denotes the Fourier transform with the convention $\widehat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx$):

Lemma 3.2 *Suppose that ρ_0 and κ are constant, and that $\Omega = \mathbb{R}^3$.*
(i) Assume that $V(t,x) = \alpha\delta(t-T)$ and that the initial conditions verify $(1+|\xi|^3)\widehat{p}_0(\xi) \in L^1(\mathbb{R}^3)$ and $(1+|\xi|^3)\widehat{\mathbf{v}}_0(\xi) \in (L^1(\mathbb{R}^3))^3$. Then, $\partial_t \nabla p \in (L^\infty(\mathbb{R}^+ \times \mathbb{R}^3))^3$.
(ii) Assume now that $V(t,x) = \alpha\eta_\tau(t-T)$, with $\eta_\tau(t) = \tau^{-1}$ on $(T-\tau/2, T+\tau/2)$ and zero otherwise. Decomposing the remainder $\mathbf{R}^\tau$ as $\mathbf{R}^\tau = (\mathbf{E}^\tau, w^\tau)$, there is a constant C such that

$$\sup_{t\in\mathbb{R}^+,\, x\in\mathbb{R}^3} \big(|\partial_t \mathbf{E}^\tau(t,x)| + |w^\tau(t,x)|\big) \leq \alpha\tau C\Big(\||\xi|^3\widehat{p}_0\|_{L^1} + \||\xi|^3\widehat{\mathbf{v}}_0\|_{L^1}\Big). \tag{10}$$

The proof of the lemma is postponed to section 5.2. The term in the parentheses in the r.h.s. above measures the spatial frequencies in the initial condition. When τ is small compared to these frequencies (up to a multiplicative dimensional factor), then the term $\mathbf{R}^\tau$ is negligible. The proof of Lemma 3.2 is based on the Fourier transform. An extension to more general settings with spatially varying coefficients would require a (non-trivial yet quite standard) microlocal analysis of (1) that is beyond the scope of this work. The proof uses analytical expressions of the unperturbed and perturbed solutions P and p.

The refocused signal. We now concentrate on the main term of the expansion given by $\mathbf{u}_R(2T)$. We recall that the matrix Γ is defined by $\Gamma = $Diag$(-1,-1,-1,1)$. We list a number of properties.

(i) When $\eta_\tau(s) = \delta(s)$, we obtain $\mathbf{u}_R(2T) = -\frac{\alpha}{2}\Gamma\partial_t\mathbf{U}(0)$ and the refocusing is 'perfect' in the sense that $\partial_t\mathbf{U}(0)$ is recovered exactly up to a multiplication factor. This makes explicit the 'differentiation' operator $\mathcal{D}$ of the introduction.

(ii) When η_τ is not a Dirac function, then waves are reversed at different times. Refocusing is then *blurred* and given by a weighted superposition of $\partial_t\mathbf{U}(2s)$ around $s=0$. The term $\mathbf{u}_R(2T)$ can be written as

$$\begin{aligned} \mathbf{u}_R(2T,x) &= \int_\Omega \mathbf{K}^\tau(x,y)\boldsymbol{\phi}_0(y)dy \\ \mathbf{K}^\tau(x,y) &= -\frac{\alpha}{2}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \eta_\tau(s)\Gamma\mathbf{G}(2s,x,y)ds. \end{aligned}$$

The kernel $\mathbf{K}^\tau$ is an approximation of $-\alpha\delta(x-y)\Gamma/2$ when τ is sufficiently small since $\mathbf{G}(0,x,y) = \delta(x-y)I$. When τ is not small, then no interesting refocusing emerges.

(iii) When ρ_0 and κ are constant and when $\Omega = \mathbb{R}^3$, we can obtain a more precise expression for $\mathbf{u}_R(2T)$. Indeed, we have in that case

$$\partial_t P(t) = \partial_t^2 G_t(p_0) + \partial_t G_t(p_1),$$

where $p_1 = -\kappa^{-1}\nabla\cdot\mathbf{v}_0 = \partial_t P(t=0)$. Above, we used notation (7) with $\mathbf{G}$ replaced by G, where G is the Green's function of the wave equation with constant coefficients. For $c_0^2 = (\rho_0\kappa)^{-1}$, we have $G(t,x,y) = G(t,x-y)$ and its Fourier transform reads

$$\widehat{G}(t,\xi) = \frac{\sin c_0 t|\xi|}{c_0|\xi|}.$$

Then, the fourth component of $\mathbf{u}_R(2T)$ (the pressure part), is

$$(\mathbf{u}_R(2T))_4 = F_0 \star p_0 + F_1 \star p_1, \tag{11}$$

where the filters F_0 and F_1 are defined by

$$\widehat{F}_0(\xi) = \frac{\alpha c_0|\xi|}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \eta(s)\sin(2c_0\tau s|\xi|)ds$$
$$\widehat{F}_1(\xi) = -\frac{\alpha}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \eta(s)\cos(2c_0\tau s|\xi|)ds.$$

When η is even, $\widehat{F}_0 = 0$ and $\widehat{F}_1$ is given by

$$\hat{F}_1(\xi) = -\frac{1}{2}\alpha\widehat{\eta}(2c_0\tau|\xi|). \tag{12}$$

The above formula and (11) show that the refocusing signal is a low-pass filter of the initial signal with a bandwidth proportional to τ^{-1}. If M is the largest wavenumber in u_0, we thus obtain a good reconstruction of the original signal when $2c_0\tau M \ll 1$. In that case, $F_1(x) \simeq -\frac{\alpha}{2}\delta(x)$.

A similar calculation using

$$\partial_t\mathbf{V}(t) = -\rho_0^{-1}\nabla P(t) = -\rho_0^{-1}\nabla\big(\partial_t G_t(p_0) + G_t(p_1)\big),$$

shows that the velocity part of the refocused signal $\mathbf{u}_R(2T)$ (i.e. the first three components) reads

$$-F_1 \star (\partial_t\mathbf{V}(t=0)) + F_2 \star (\mathbf{V}(t=0)),$$

with $\widehat{F}_2(\xi)\widehat{\mathbf{v}}_0(\xi) = \widehat{F}_0(\xi)\hat{\xi}(\hat{\xi}\cdot\widehat{\mathbf{v}}_0(\xi))$ and $\hat{\xi} = \xi/|\xi|$. When η is not even, the kernel F_0 does not vanish, and therefore the term p_0 contributes to the pressure part of $\mathbf{u}_R(2T)$ and $\mathbf{v}_0$ to the velocity part. We will explore this fact further numerically. When $2c_0\tau M \ll 1$, these contributions are nevertheless small, although they increase with τ.

Multiple perturbations. We now briefly consider the case of N_p successive time-dependent perturbations at multiples of T, that is

$$V(t) = \alpha\sum_{k=1}^{N_p}\eta_\tau(t-kT).$$

For $k=1$, the form of the refocused signal at $t=2T$ is given by Theorem 3.1. After the first perturbation, there are two propagating waves: the unperturbed wave $\mathbf{U}$, and the forward wave created by the perturbation $\mathbf{u}_F = \alpha\partial_t\mathbf{U}/2$. Denote their sum by $\mathbf{u}_p$. The second perturbation creates a refocused signal at $t=4T$ equal to

$$-\frac{\alpha}{2}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\eta_\tau(s)\Gamma\partial_t\mathbf{u}_p(2s)ds,$$

with a total propagating wave for $t \geq 2T+\tau/2$ equal to $\mathbf{u}_p+\alpha\partial_t\mathbf{u}_p/2$. Repeating the procedure, we find that the refocused signal at $t=kT$ is a linear combination of terms of the form

$$\left(\frac{\alpha}{2}\right)^k\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\eta_\tau(s)\Gamma\partial_t^{(k)}\mathbf{U}(2s)ds.$$

In particular, the pressure component of $\partial_t^{(k)}\mathbf{U}$ verifies

$$\partial_t^{(k)}P(t) = \partial_t^{(k+1)}G_t(p_0) + \partial_t^{(k)}G_t(p_1).$$

Setting for instance $k = 2$ and $p_1 = 0$, we have, when the background velocity $c = c_0$ is constant,

$$\partial_t^{(3)} G_t(p_0) = \partial_t^{(2)} P(t) = c_0^2 \partial_t G_t(\Delta p_0), \qquad \text{and} \qquad \partial_t P(t) = c_0^2 G_t(\Delta p_0).$$

Since $\partial_t G(t,x,y) \simeq \delta(x-y)$ and $G(t,x,y) \simeq 0$ for $t \simeq 0$, the refocused signal is then predominantly a mollified version of the Laplacian of the initial condition p_0. We will verify this numerically in the next paragraph (see Fig. 3). More generally, if c_0 is constant and if $k = 2m$, then

$$\begin{aligned} \partial_t^{(2m)} P(t) &= c_0^{2m} \partial_t G_t(\Delta^{(m)} p_0) + c_0^{2m} G_t(\Delta^{(m)} p_1) \\ &\simeq c_0^{2m} \Delta^{(m)} p_0 \quad \text{for } t \simeq 0 \quad \text{if} \quad p_0 \neq 0, \end{aligned}$$

while if $k = 2m + 1$,

$$\begin{aligned} \partial_t^{(2m+1)} P(t) &= c_0^{2(m+1)} G_t(\Delta^{(m+1)} p_0) + c_0^{2m} \partial_t G_t(\Delta^{(m)} p_1) \\ &\simeq c_0^{2m} \Delta^{(m)} p_1 \quad \text{for } t \simeq 0 \quad \text{if} \quad p_1 \neq 0. \end{aligned}$$

To conclude: when the background is constant, multiple perturbations produce time-reversed signals proportional to $\partial_t^{(k)} \mathbf{U}$, in which, according the equations on $\partial_t^{(k)} P$ above, the pressure component is proportional to powers of the Laplacian of the initial condition.

The rest of the section is dedicated to numerical simulations.

Simulations. We solve the wave equation (4) on a square of side 2 centered at the origin. The equation is equipped with Dirichlet boundary conditions for simplicity. We set $T = 1.8$, and the discretization step sizes are $\Delta x = 0.002$ and $h := \Delta t = 0.00072$. The initial conditions are centered around the origin, and the background coefficients ρ_0 and κ are constant and such that $c(x) = c_0 = 1$.

In figure 1, we investigate the blurring effect as τ increases. We set $p_0 = 0$ and p_1 a smiley figure, and represent $p(2T)$ for several values of τ. The function η_τ is modeled numerically by a constant function over $n_\tau + 1$ time steps with integral one. The strength of the perturbation is $\alpha = 0.001$. We clearly observe the blurring effect as n_τ increases.

In figure 2, we investigate the effect of the evenness of the function η_τ, and have seen theoretically (in the case $\Omega = \mathbb{R}^3$) that the contribution of p_0 to the refocused signal is zero when η_τ is even. We keep the same p_1 as before and set p_0 as the function represented on the left panel (the sum of four Gaussians). The middle figure is the refocused signal for a constant function η_τ over 13 time steps, and we observe numerical effects creating a non zero contribution of p_0. As expected, the latter is strongly increased when the function η_τ is an uneven piecewise constant function as observed on the right panel (η_τ takes the value $4/C$ on the first 3 time intervals and $1/C$ on the others, for C an appropriate normalization constant).

In figure 3, we verify that after two perturbations at $t = T$ and $t = 2T$ and with initial conditions p_0 and $p_1 = 0$, the refocused signal at $t = 4T$ is proportional to the Laplacian of p_0.

We consider in the next section spatially dependent perturbations and compare with the spatially independent case.

4 Spatially-dependent perturbation

We suppose in this section that $\Omega = \mathbb{R}^3$ and that the time-dependent perturbation is localized in space, that is $V(t,x) = \alpha \eta_\tau(t-T)\chi(x)$, where $\chi(x)$ is a (non-negative) function with

Figure 1: Refocused signal for p_0=0 and p_1 a smiley figure. Top left: perturbation of duration h. Top right: perturbation of duration $5h$. Bottom left: perturbation of duration $7h$. Bottom right: perturbation of duration $13h$. The blurring becomes apparent as the duration of the perturbation increases.

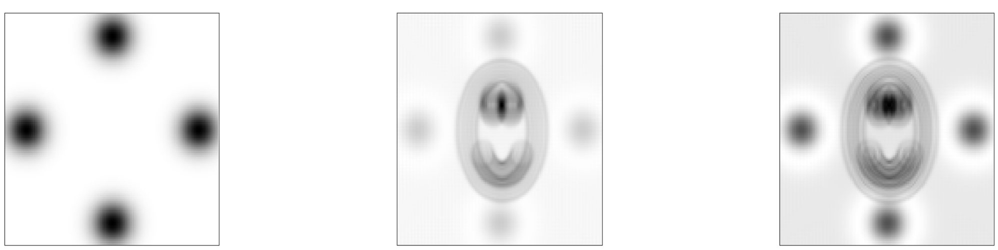

Figure 2: Refocused signal for p_1 as in figure 1 and p_0 given by the function represented on the left panel. Left: initial condition p_0. Middle: perturbation of duration $13h$ for a constant function η_τ. Right: perturbation of duration $13h$ for an uneven function η_τ. The strength of the contribution of p_0 increases when η_τ is not even.

bounded support. As mentioned in the introduction, we wish to consider an experimental setting for time reversal that (i) does not necessitate the heavy antennas/transducers apparatus and memory requirements of *classical* time-reversal; and (ii) assumes that the time-dependent fluctuations can be performed only on (a small) part of the spatial domain.

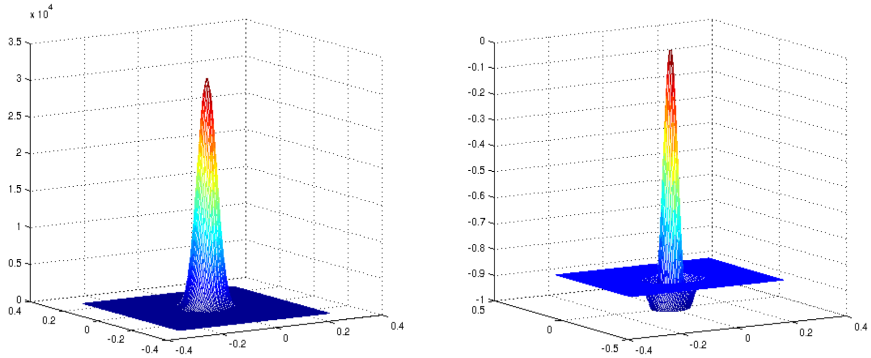

Figure 3: Refocused signal after two perturbations of duration h. Left: a Gaussian initial condition for p_0. Right: refocused signal at $4T$. We verify that the latter is indeed proportional to the laplacian of p_0.

We consider two situations: (i) homogeneous media, and in such a case only the waves on the support of χ are reversed and the quality of refocusing will be limited by classical diffraction; (ii) highly heterogeneous media (modeled here by random media with appropriate statistics), with a much enhanced resolution as in the classical theory of time reversal. The main result of the section is (the somewhat informal) Theorem 4.1 below. It provides a quantitative description of the time-reversal experiment as a band-pass filter and explains why refocusing is much enhanced when propagation occurs in a rich multi-path environment.

Scalings. We suppose that the initial condition $(\mathbf{v}_0, p_0)$ is centered at x_0 with support of order λ. Equivalently, the quantity $2\pi/\lambda$ can be seen as the largest spatial frequency. We suppose that the sound speed $c^2(x) = (\rho_0(x)\kappa(x))^{-1}$ verifies

$$c_- \le c(x) \le c_+,$$

for two positive constants c_- and c_+. For T the time of the perturbation, let $L = c_- T$, which is the minimum distance traveled by the wave at time T. We then assume that $L \gg \lambda$, and introduce

$$\varepsilon = \frac{\lambda}{L} \ll 1.$$

Rescaling all position variables by L, we rewrite $\boldsymbol{\phi}_0 = \partial_t \mathbf{U}(t=0) = -(\rho_0^{-1}\nabla p_0, \kappa^{-1}\nabla\cdot\mathbf{V}_0)$ as

$$\boldsymbol{\phi}_0(Lx') = \frac{1}{\varepsilon^3}\mathbf{Q}\left(\frac{x'-x'_0}{\varepsilon}\right), \qquad x_0 = Lx'_0.$$

We model the heterogeneous medium by a random medium and assume the following forms for the unperturbed (w.r.t. time) density and compressibility:

$$\rho_0(x) = \bar\rho(x)\left(1+\sigma_\rho\theta_\rho\left(\frac{x}{\ell_c}\right)\right), \qquad \kappa(x) = \bar\kappa(x)\left(1+\sigma_\kappa\theta_\kappa\left(\frac{x}{\ell_c}\right)\right),$$

where θ_ρ and θ_κ are mean-zero stationary random fields with correlation functions R_ρ and R_κ. The parameters σ_ρ and σ_κ quantify the strength of the fluctuations and ℓ_c their correlation length. Introducing $\delta = \ell_c/L$ and rescaling x by Lx', ρ_0 and κ become $\rho_0(Lx') = \bar\rho(Lx')(1+\sigma_\rho\theta_\rho(x'/\delta))$, and $\kappa(Lx') = \bar\kappa(Lx')(1+\sigma_\kappa\theta_\kappa(x'/\delta))$. There are essentially three interesting asymptotic regimes depending on the relationship between $\sigma_\rho, \sigma_\kappa, \delta$ and ε. In the *stochastic* *homogenization regime*, $\sigma_\rho \sim \sigma_\kappa \sim 1$, and $\delta \ll \varepsilon$. In this situation, waves are asymptotically propagating in an effective medium and no significant mixing occurs. It is therefore expected that refocusing will be similar to the homogeneous case, and as a consequence this regime will not be considered. The opposite situation where $\varepsilon \ll \delta$, with σ_ρ and σ_κ small (the appropriate relation with ε can be found in [3]), is called the *random geometrical optics regime*, and exhibits some mixing leading to super-resolution. The intermediate case, the *radiative transfer regime* where $\sigma_\rho = \sigma_\kappa = \sqrt{\varepsilon}$ and $\delta = \varepsilon$, leads to some mixing as well. We study then the time-reversed wave in these last two regimes.

Asymptotic analysis. Let $\mathbf{u}_R^\varepsilon(\xi) = \mathbf{u}_R(2T, L(x_0 + \varepsilon\xi))$. We nondimensionalize time and position as $t \to Tt'$ and $x \to Lx'$ (so that the perturbation occurs at $t' = 1$ in nondimensional variables), drop primes, and keep the same notation for the various quantities of interest. The new matrix A is then the old one multiplied by c_-, and the old Green's functions are divided by L^3. We also introduce $\mu_\varepsilon = \tau/(\varepsilon T)$, which we assume converges to $\mu \in [0, \infty]$ as $\varepsilon \to 0$. We will adapt the theory of time-reversal introduced in [5, 6] and for this we need to appropriately decompose the term $\mathbf{Q}$. Let then

$$\mathbf{b}^\pm(x,k) = \frac{1}{\sqrt{c_-}} \begin{pmatrix} \pm\hat{k}/\sqrt{2\bar{\rho}(x)} \\ 1/\sqrt{2\bar{\kappa}(x)} \end{pmatrix}, \qquad \hat{k} = \frac{k}{|k|},$$

which are the two eigenvectors of the dispersion matrix $\mathbb{E}\{A(x)\}^{-1}k_j D^j$ associated with the simple eigenvalues $\pm\omega(x,k)$ with $\omega(x,k) = (\bar{c}(x)/c_-)|k|$ and $\bar{c}(x) = (\bar{\rho}(x)\bar{\kappa}(x))^{-1/2}$. They are normalized such that $\mathbf{b}^\pm \cdot \mathbb{E}\{A(x)\}\mathbf{b}^\pm = 1$. The other (double) eigenvalue is zero. Since the first three components of $\mathbf{Q}$ form an irrotational vector field, $\mathbf{Q}$ has no contributions on the eigenspace associated with the zero eigenvalue, and its Fourier transform can then be decomposed as

$$\widehat{\mathbf{Q}}(k) = \widehat{\mathbf{Q}}^+(k) + \widehat{\mathbf{Q}}^-(k) := Q^+(k)\mathbf{b}^+(x_0,k) + Q^-(k)\mathbf{b}^-(x_0,k). \tag{13}$$

The decomposition of Lemma 2.1 still holds here, with the difference that, in the heterogeneous case, the contributions of the unperturbed solution $\mathbf{U}$ and the forward solution $\mathbf{u}_F$ to $\mathbf{u}(2T)$ around the origin are non-zero due to multiple scattering. This brings in some background noise in the reconstructions. Regarding the time-reversed wave, we have the following (informal) result:

Theorem 4.1 *The refocused signal $\mathbf{u}_R^\varepsilon$ verifies*

$$\lim_{\varepsilon\to 0} \mathbf{u}_R^\varepsilon(\xi) = (\mathcal{J}^+ \star \mathbf{Q}^+)(\xi) + (\mathcal{J}^- \star \mathbf{Q}^-)(\xi),$$

where

$$\widehat{\mathcal{J}^\pm}(k) = -\frac{\alpha}{2}\hat{\eta}\big(\pm 2\mu\,\omega(x_0,k)\big)a_\pm(1,x_0,k)\Gamma.$$

Above, $a_\pm$ is the solution to the transport equation

$$\begin{cases} \frac{\partial a_\pm}{\partial t} \pm \nabla_k\omega \cdot \nabla_x a_\pm \mp \nabla_x\omega \cdot \nabla_k a_\pm = \mathcal{L}(a_\pm), & (x,k) \in \mathbb{R}^3 \times \mathbb{R}^3 \\ a_\pm(0,x,k) = \chi(x), \end{cases} \tag{14}$$

where the operator $\mathcal{L}$ admits the following expressions:

$$\mathcal{L}(a)(x,k) = \begin{cases} \displaystyle\int_{\mathbb{R}^3} \delta(\omega(x,p)-\omega(x,k))\sigma(x,k,p)(a(x,p)-a(x,k))dp \\ \qquad\qquad\qquad \text{in the radiative transfer regime,} \\ \displaystyle\frac{\partial}{\partial k_m}\left(|k|^2 D_{jm}(x,\hat{k})\frac{\partial a(x,k)}{\partial k_j}\right) \\ \qquad\qquad\qquad \text{in the random geometrical optics regime.} \end{cases}$$

The definitions of the collision cross-section σ and of the diffusion matrix D above depend on the Fourier transforms of $\mathbb{E}\{\theta_\rho(x)\theta_\rho(0)\}$, $\mathbb{E}\{\theta_\kappa(x)\theta_\kappa(0)\}$ and $\mathbb{E}\{\theta_\rho(x)\theta_\kappa(0)\}$, and can be found in [3, 19].

Theorem 4.1 shows that, as in the classical theory of time reversal [6], the refocused signal is the convolution between some filters $\mathcal{J}^\pm$ and an appropriate quantity (here $\partial_t \mathbf{U}(t=0)$, while it is $\mathbf{U}(t=0)$ in [6], both after appropriate projections). For good reconstructions, the kernels $\mathcal{J}^\pm$ have to decrease rapidly away from the origin, which is equivalent to their Fourier transforms $\widehat{\mathcal{J}^\pm}$ being smooth functions. The theorem is specialized to the three dimensional case for simplicity, but holds in any dimension $d \geq 2$. It does not hold for $d=1$ as radiative transfer equations are not valid.

Enhanced refocusing in heterogeneous media is explained as follows. Consider first the term $\hat{\eta}\big(\pm 2\mu\,\omega(x_0,k)\big)$, which is smooth. Indeed, as soon as $|k|>0$, $\omega(x_0,k)$ is a smooth function (zero frequency waves for $k=0$ do not propagate and are therefore not reversed), and $\hat{\eta}$ is smooth as well since η has a compact support. The key parameter is then μ: when $\mu=\infty$, that is essentially when $\tau \gg \varepsilon T$, then $\mathcal{J}^\pm = 0$ since $\hat{\eta}$ is zero at infinity by applying the Riemann-Lebesgue lemma. There is no refocused wave in this case since the duration of the perturbation is too large and waves interfere destructively. When μ is finite, the best refocusing is obtained for $\tau|k| \ll \varepsilon T$, that is when the duration of the perturbation is sufficiently small compared to the largest wavenumber in the initial condition. This is the same scenario as in the case of a spatially independent perturbation addressed in section 3. The difference with the latter lies in the term $a_\pm$. The fact that $a_\pm$ is more regular in the heterogeneous case than in the homogeneous case is explained in the same way as in the classical theory of time reversal developed in [6]: in the homogeneous case where $\mathcal{L}=0$, $a_\pm$ is essentially as regular as χ, while in the heterogeneous case, the transport equation (14) enjoys some regularizing effects.

In the random geometrical optics regime, the transport operator is hypoelliptic, while in the radiative transfer regime, the solution $a_\pm$ can be written as a multiple scattering expansion in which each term decays exponentially (reducing in particular exponentially the contribution of the singular term χ) and where the $(n+1)$-th term is more regular than the $n-$th term, see [6] for more details. This explains the super-resolution phenomenon in the heterogeneous case.

The proof of Theorem 4.1 is given in section 5.3. It is based on an adaptation of the techniques of [6] and on the introduction of space-time Wigner transforms.

Note that when $\chi=1$, then $a_\pm=1$ and we recover the results for the spatially independent case as follows: writing first

$$\begin{aligned} \hat{\eta}\big(\pm 2\mu\,\omega(x_0,k)\big) &= \int_{\mathbb{R}} \eta(s)\cos(2\mu\,\omega(x_0,k)s)ds \mp i\int_{\mathbb{R}} \eta(s)\sin(2\mu\,\omega(x_0,k)s)ds \\ &:= \hat{a}(k) \pm i\hat{b}(k), \end{aligned}$$

we find from Theorem 4.1,

$$\lim_{\varepsilon\to 0}\Gamma \mathbf{u}_R^\varepsilon(\xi) = -\frac{\alpha}{2} a \star \mathbf{Q} - \frac{\alpha}{2} ib \star (\mathbf{Q}^+ - \mathbf{Q}^-). \tag{15}$$

A short calculation shows that $\widehat{\mathbf{Q}}^+ - \widehat{\mathbf{Q}}^-$ is equal to the vector $-i(|\xi|\,\hat{\xi}\cdot\mathbf{v}_0, |\xi|\,\hat{p}_0)$. We then compare with Theorem 3.1: rescaling variables and the initial condition as in this section, we have that $\partial_t \mathbf{U}(x_0+\varepsilon\xi)$ is well approximated for ε small by the solution to the wave equation with frozen coefficients $\bar{\kappa}(x_0)$, $\bar{\rho}(x_0)$ and with the same (rescaled) initial conditions. It suffices then to reproduce the analysis in the paragraph after Lemma 3.2 to recover expression (15).

Simulations. We choose χ to be the smoothed characteristic function of disks centered at $(0.35, 0.35)$ with radii 0.1 and 0.3, see figure 4. The initial condition is centered at zero as before. We solve the wave equation (4) both when the background is constant and when it is random (heterogeneous).

We start with the homogeneous case and represent in figure 5 the refocused signal for perturbations on the small and large disks with $p_0 = 0$ and p_1 a smiley figure. We set $T = 0.5$ so that the wavefront is at the center of the disks. We choose $n_\tau = 0$ (recall that $n_\tau + 1$ is the number of time steps of the perturbation) and the perturbation is then close to a delta function. The smiley figure can barely be recognized in the small disk case and refocusing is very poor. It is better for the large disk, as expected. We investigate this fact further in figure 6 where we set $p_0 = 0$ and p_1 as a peaked Gaussian in order to characterize the point-spread function. The resolution in the direction of the disks is good (range resolution), while the cross-range resolution is poor in the case of the small disk, and naturally better for the larger disk.

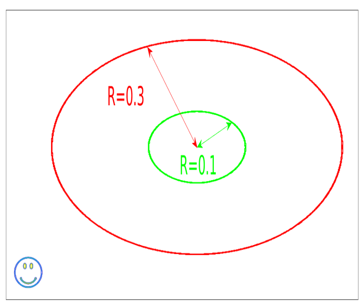

Figure 4: Depiction of the initial condition and the support of the function χ, which is the smoothed characteristic function of either the small or the large disk where the waves are perturbed.

We consider now the heterogeneous case with two random media with fluctuations with different correlation lengths and similar amplitudes, see figure 7. The one on the left is very rough and essentially spatial white noise, and corresponds to the radiative transfer regime as our initial condition (the smiley figure) is very singular. The one on the right is smoother with a larger correlation length, and corresponds to the random geometrical optics regime. Both have fluctuations of about the same amplitude and are generated using random Fourier series. The power spectrum of the fluctuations is the characteristic function of a disk centered at zero with radius k_m. In the rough medium case, $k_m = 2\pi \times \frac{4}{5\Delta x}$ (and hence fluctuations occur independently at a scale close to the grid step), while we have $k_m = 2\pi \times \frac{1}{5\Delta x}$ in the smooth case.

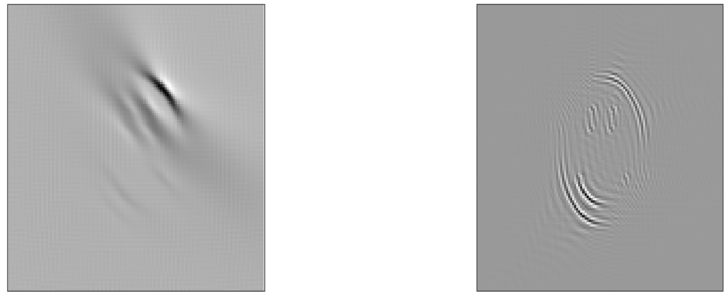

Figure 5: Refocused signal in the homogeneous case ($p_0 = 0$). Left: small disk. Right: large disk. The reconstruction is better for the larger disk as expected.

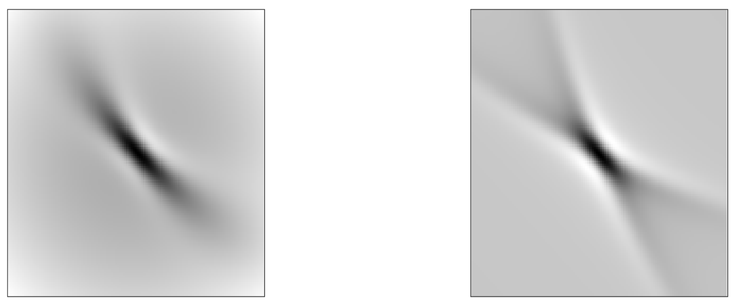

Figure 6: Point spread function in the homogeneous case. Left: small disk. Right: large disk. The large disk offers a better cross-range resolution.

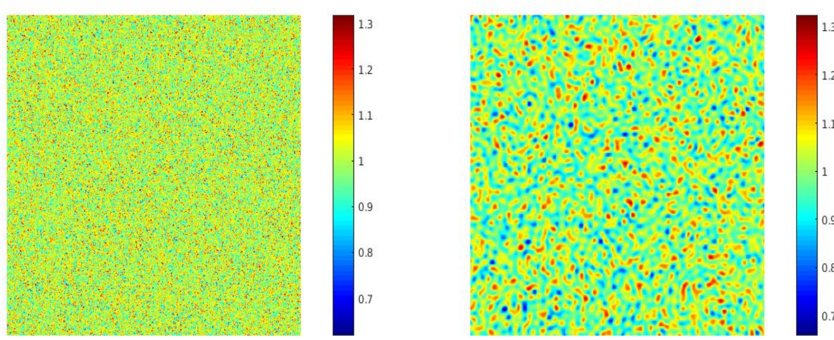

Figure 7: Random media: background (i.e. average value) is one, with mean-zero random perturbations. Left: rough medium close to spatial white noise. Right: smoother medium with larger correlation length. The amplitudes of the fluctuations in both media are similar.

We represent in figure 8 the refocused signal for the two random media and the two disks of different radii. The time T is set to 1.8 for sufficient interaction between the wave and medium, and the strength of the perturbation is increased to $\alpha = 0.01$ in order to increase the signal-to-noise ratio (SNR). The signal is here the refocused signal $\mathbf{u}_R$, and the noise corresponds to the other terms in the decomposition (8), namely $\mathbf{U}+\mathbf{u}_F+\mathbf{R}^\tau$. We do not evaluate quantitatively the SNR, but instead observe the constrast between the signal and noise to obtain a qualitative estimate. The top row corresponds to the smooth random medium that is on the right of figure 7, while the bottom row corresponds to the rough medium. The left column corresponds to the small disk, and the right one to the large disk.

Refocusing is spectacularly improved in the case of the rough medium compared to the homogeneous case, both for the small and the large disks. Apart from the noise in the background, the reconstruction of the initial condition is almost perfect. Refocusing is enhanced in the smooth medium case compared to the homogeneous situation, but not as much as in the rough medium case. This is explained by the very singular nature of our initial condition, which better interacts with the rough random medium than with the smooth one, resulting in more mixing at the time of the perturbation. This fact is further explored in figure 9, where we represent the point spread function (calculated for $T = 1.8$ with $p_0 = 0$ and p_1 a peaked Gaussian). Again, the large disk and the rough medium offer the best resolution and the best SNR. The SNR decreases in the case of the small disk, but, when the medium is rough, the resolution is comparable to that of the large disk. When the medium is smooth, the central peak is wider than in the rough case, and can barely be observed when the disk is small.

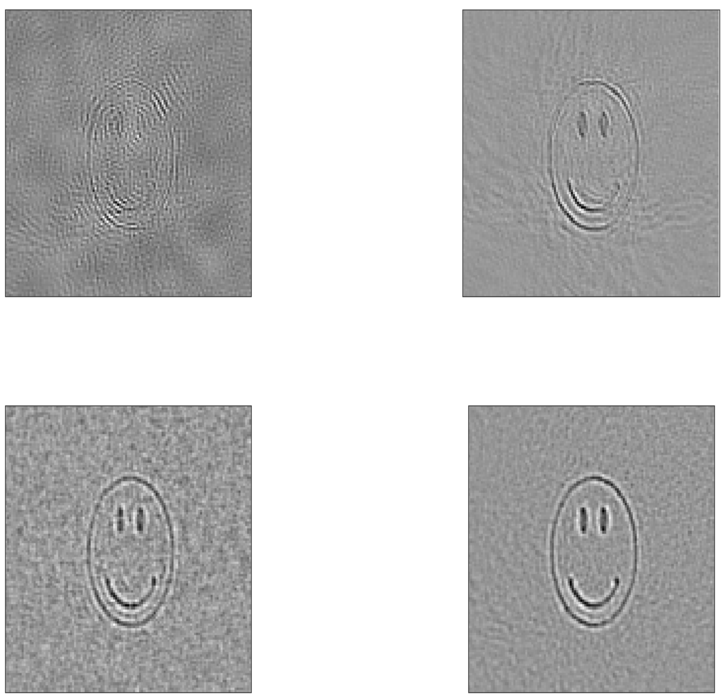

Figure 8: Refocused signal in random media (with $p_0 = 0$). Top row: smooth random medium. Bottom row: rough random medium. Left column: small disk. Right column: large disk. For the rough random medium, note the much better refocusing than in the homogeneous case.

The rest of the paper is dedicated to the proofs of our main results.

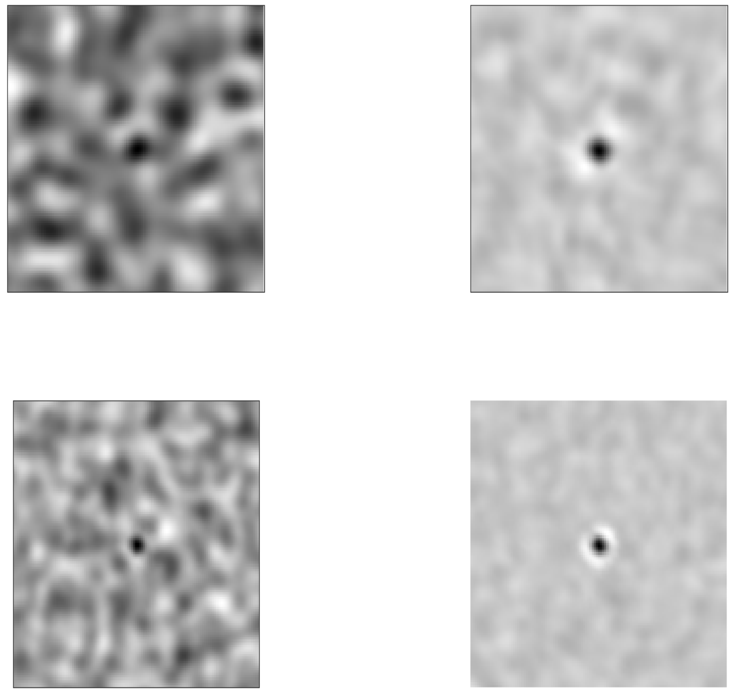

Figure 9: Point spread function. Top row: smooth random medium. Bottom row: rough random medium. Left column: small disk. Right column: large disk. The best resolution and best signal-to-noise ratio are observed in the rough medium and large disk case.

5 Proofs

5.1 Proof of Lemma 2.1

Introducing the Green's function $\mathbf{G}$, we have the representation formula

$$\mathbf{u}(t) = \mathbf{G}_t(\mathbf{u}_0) + \int_0^t \mathbf{G}_{t-s}(V_s A^{-1}\mathbf{S}_s)ds = \mathbf{U}(t) + \int_0^t \mathbf{G}_{t-s}(V_s A^{-1}\mathbf{S}_s^0)ds + \mathbf{R}^\tau(t) \tag{16}$$

where

$$\mathbf{R}^\tau(t) = \int_0^t \mathbf{G}_{t-s}(V_s A^{-1}(\mathbf{S}_s - \mathbf{S}_s^0))ds \qquad \text{with} \qquad \mathbf{S}^0(t) = -(\nabla P(t), 0) = \rho_0(\partial_t \mathbf{V}(t), 0),$$

and we recall $(\mathbf{V}, \mathbf{U})$ is the unperturbed solution. With the (column) vector $\partial_t \mathbf{U}(t) = (\partial_t \mathbf{V}(t), \partial_t P(t))$ and the matrix $\Gamma = \mathrm{Diag}(-1, -1, -1, 1)$, the second term of the r.h.s. of (16) above is recast as

$$-\frac{1}{2}\int_0^t \mathbf{G}_{t-s}(V_s \Gamma \partial_t \mathbf{U}_s)ds + \frac{1}{2}\int_0^t \mathbf{G}_{t-s}(V_s \partial_t \mathbf{U}_s)ds.$$

We call the first term above $\mathbf{u}_R$ and the second one $\mathbf{u}_F$. The vector $\partial_t \mathbf{U}$ is the solution to the system (5) with $V = 0$ and initial condition $\partial_t \mathbf{U}(t = 0) = \boldsymbol{\phi}_0 = -(\rho_0^{-1}\nabla p_0, \kappa^{-1}\nabla \cdot \mathbf{v}_0)$. As a consequence, the term $\mathbf{u}_R$ can then be written as

$$\begin{aligned}\mathbf{u}_R(t) &= -\frac{\alpha}{2}\int_0^t \eta_\tau(s-T)\mathbf{G}_{t-s}(\chi\Gamma\mathbf{G}_s(\boldsymbol{\phi}_0))ds \\ &= -\frac{\alpha}{2}\int_0^t\int_\Omega\int_\Omega \eta_\tau(s-T)\mathbf{G}_{t-s}(x,y)\Gamma\mathbf{G}_s(y,z)\chi(y)\boldsymbol{\phi}_0(z)dsdydz,\end{aligned}$$

which ends the proof.

5.2 Proof of Lemma 3.2

We start by showing that $\partial_t\nabla p\in(L^\infty(\mathbb{R}^+\times\mathbb{R}^3))^3$ when $V(t,x)=\alpha\delta(t-T)$, assuming the initial conditions are sufficiently smooth. Taking the Fourier transform of the wave equation yields

$$\partial_t^2\widehat{p}+c_0^2|\xi|^2\big(1+\alpha\delta(t-T)\big)\widehat{p}=0,\qquad \widehat{p}(t=0)=\widehat{p}_0,\qquad \partial_t\widehat{p}(t=0)=\widehat{p}_1.$$

The equation above can be solved exactly by imposing that $\widehat{p}$ is continuous and that $\partial_t\widehat{p}$ jumps at $t=T$. We find:

$$\widehat{p}(t,\xi)=\begin{cases}\widehat{p}_0(\xi)\cos\big(c_0|\xi|t\big)+\widehat{p}_1(\xi)\sin\big(c_0|\xi|t\big)/(c_0|\xi|) & \text{for}\quad t\le T\\ \widehat{p}_0(\xi)F_1(t,\xi)+\widehat{p}_1(\xi)F_2(t,\xi) & \text{for}\quad t\ge T,\end{cases}$$

where, using the shorthand $\omega_T=c_0|\xi|T$,

$$\begin{aligned}F_1(t,\xi)&=(1+\alpha c_0|\xi|\cos\omega_T\sin\omega_T)\cos(c_0|\xi|t)-\alpha c_0|\xi|\cos^2\omega_T\sin(c_0|\xi|t)\\ F_2(t,\xi)&=\left(\frac{1}{c_0|\xi|}-\alpha\cos\omega_T\sin\omega_T\right)\sin(c_0|\xi|t)+\alpha\sin^2\omega_T\cos(c_0|\xi|t).\end{aligned}$$

The explicit representation of $\widehat{p}$ then allows us to derive the following estimate, for all $t\neq T$,

$$|\partial_t\widehat{p}(t,\xi)|\le C|\xi|(1+|\xi|)|\widehat{p}_0(\xi)|+C'(1+|\xi|)|\widehat{p}_1(\xi)|.$$

Since $\|\partial_t\nabla p\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)}\le\||\xi|\partial_t\widehat{p}\|_{L^\infty(\mathbb{R}^+,L^1(\mathbb{R}^3))}$, the result is obtained provided $(1+|\xi|^3)\widehat{p}_0(\xi)\in L^1(\mathbb{R}^3)$ and $(1+|\xi|^2)\widehat{p}_1(\xi)\in L^1(\mathbb{R}^3)$.

We prove now the estimate on the remainder $\mathbf{R}^\tau$. For this, let w be the fourth component of $\mathbf{R}^\tau$. It satisfies the equation

$$\partial_t^2 w-c_0^2\Delta w=\alpha\eta_\tau(t)c_0^2\Delta(p-P),\qquad w(t=0)=\partial_t w(t=0)=0,$$

for p and P the perturbed and unperturbed pressures, respectively. We remark first that $w=0$ when $t\le T-\frac{\tau}{2}$ and that for $t\ge T-\frac{\tau}{2}$, w admits the representation formula in the Fourier space:

$$\widehat{w}(t,\xi)=-\alpha c_0|\xi|\int_{-\frac{\tau}{2}}^{t\wedge\frac{\tau}{2}}\eta_\tau(s)\sin\big(c_0(t-s-T)|\xi|\big)\big(\widehat{p}(s+T,\xi)-\widehat{P}(s+T,\xi)\big)ds.$$

Since η_τ integrates to one, we have then, for all $t\ge T-\frac{\tau}{2}$,

$$|\widehat{w}(t,\xi)|\le\alpha c_0|\xi|\sup_{t\in[T-\frac{\tau}{2},T+\frac{\tau}{2}]}|\widehat{p}(t,\xi)-\widehat{P}(t,\xi)|. \tag{17}$$

It therefore remains to control $\widehat{p}-\widehat{P}$ on $[T-\frac{\tau}{2},T+\frac{\tau}{2}]$. This is done as follows. First, $\widehat{p}$ verifies

$$\Big(\partial_t^2+c_0^2|\xi|^2(1+\alpha\eta_\tau(t))\Big)\widehat{p}=0,\qquad \widehat{p}(t=0)=\widehat{p}_0,\quad \partial_t\widehat{p}(t=0)=\widehat{p}_1.$$

Since η_τ is constant on the interval $[T-\tau/2, T+\tau/2]$, $\widehat{p}$ can be calculated analytically. Introducing $t_\pm = T \pm \tau/2$ and $\omega = c_0|\xi|\sqrt{1+\alpha/\tau}$, we find that $\widehat{p}$ reads, for $t \in [t_-, t_+]$,

$$\widehat{p}(t,\xi) = G_1(t,\xi)\widehat{p}_0(\xi) + G_2(t,\xi)\widehat{p}_1(\xi)$$

where

$$G_1(t,\xi) = \cos(c_0|\xi|t_-)\cos\big(\omega(t-t_-)\big) - \frac{c_0|\xi|}{\omega}\sin(c_0|\xi|t_-)\sin\big(\omega(t-t_-)\big)$$
$$G_2(t,\xi) = \frac{\sin(c_0|\xi|t_-)}{c_0|\xi|}\cos\big(\omega(t-t_-)\big) + \frac{\cos(c_0|\xi|t_-)}{\omega}\sin\big(\omega(t-t-)\big).$$

Writing then, with $\widehat{G}(t,\xi) = \sin(c_0|\xi|t)/(c_0|\xi|)$,

$$\widehat{P}(t,\xi) \quad = \quad \partial_t\widehat{G}(t,\xi)\widehat{p}_0(\xi) + \widehat{G}(t,\xi)\widehat{p}_1(\xi),$$

we need to estimate $G_1 - \partial_t\widehat{G}$ and $G_2 - \widehat{G}$ for $t \in [t_-, t_+]$. This is direct as, for all $t \in [t_-, t_+]$,

$$\begin{aligned}
|G_2(t,\xi) - \widehat{G}(t,\xi)| \quad &\leq \quad \left|\frac{\sin(c_0|\xi|t_-) - \sin(c_0|\xi|t)}{c_0|\xi|}\cos\big(\omega(t-t_-)\big)\right| \\
&\quad + \left|\frac{\sin(c_0|\xi|t)}{c_0|\xi|}\big(\cos\omega(t-t_-) - 1\big)\right| + \left|\frac{\cos(c_0|\xi|t_-)}{\omega}\sin\omega(t-t_-)\right| \\
&\leq \quad \tau + \tau(\tau+\alpha)c_0|\xi|/2 + \tau,
\end{aligned}$$

and

$$\begin{aligned}
|G_1(t,\xi) - \partial_t\widehat{G}(t,\xi)| \quad &\leq \quad \big|\big(\cos(c_0|\xi|t_-) - \cos(c_0|\xi|t)\big)\cos\big(\omega(t-t_-)\big)\big| \\
&\quad + \big|\cos(c_0|\xi|t)\big(\cos\omega(t-t_-) - 1\big)\big| + \left|\frac{c_0|\xi|}{\omega}\sin(c_0|\xi|t_-)\sin\omega(t-t_-)\right| \\
&\leq \quad c_0|\xi|\tau + \tau(\tau+\alpha)(c_0|\xi|)^2/2 + c_0|\xi|\tau.
\end{aligned}$$

Therefore,

$$\sup_{s\in[T-\frac{\tau}{2}, T+\frac{\tau}{2}]} |\widehat{p}(t,\xi) - \widehat{P}(t,\xi)| \leq \tau(2 + (\tau+\alpha)/2)(c_0|\xi||\widehat{p}_0(\xi)| + |\widehat{p}_1(\xi)|),$$

and, together with (17), we have arrived at

$$|\widehat{w}(t,\xi)| \leq \alpha\tau c_0|\xi|(2 + (\tau+\alpha)/2)(c_0|\xi||\widehat{p}_0(\xi)| + |\widehat{p}_1(\xi)|). \tag{18}$$

Representing now $\mathbf{R}^\tau$ by $\mathbf{R}^\tau = (\mathbf{E}, w)$, the vector $\mathbf{E}$ verifies

$$\partial_t\mathbf{E} = -\rho_0^{-1}\nabla w, \qquad \mathbf{E}(t=0) = \mathbf{0},$$

which allows us to directly estimate $\partial_t\mathbf{E}$ in the Fourier space in terms of w using (18). We conclude the proof of the lemma by going back to the real space.

5.3 Proof of Theorem 4.1

We remain at a formal level here, a rigorous proof of convergence of the Wigner transform in the context of random wave equations is a very difficult problem and well beyond the scope of this work, we refer to [10] for some results in this direction. With the change of variable $s \to \varepsilon s/2$ and $z \to x_0 + \varepsilon z$, we find from Lemma 2.1,

$$\mathbf{u}_R^\varepsilon(\xi) = -\frac{\alpha}{4}\int_{-\mu_\varepsilon}^{\mu_\varepsilon}\int_{\mathbb{R}^3} \eta_{\mu_\varepsilon}(s/2)\boldsymbol{\mathcal{K}}(2, \varepsilon s/2 + 1, x_0 + \varepsilon\xi, x_0 + \varepsilon z)\mathbf{Q}(z)dsdz,$$

where we recall that $\mu_\varepsilon = \tau/(\varepsilon T)$ and

$$\begin{aligned}&\boldsymbol{\mathcal{K}}(2, \varepsilon s/2+1, x_0+\varepsilon\xi, x_0+\varepsilon z)\\ &\qquad = \int_{\mathbb{R}^3} \mathbf{G}_{1-\varepsilon s/2}(x_0+\varepsilon\xi, y)\mathbf{G}^*_{1+\varepsilon s/2}(x_0+\varepsilon z, y)\Gamma A(x_0+\varepsilon z)\chi(y)dy.\end{aligned}$$

Above, we used the fact, proved in [6], that $\mathbf{G}^*(t,x,y)\Gamma A(x) = \Gamma \mathbf{G}(t,y,x)$. Introducing the space-time Wigner transform (see e.g. [2]),

$$\mathbf{W}_\varepsilon(t,\omega,x,k) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^7} e^{i(\omega s + k\cdot z)}\mathbf{G}_{t-\frac{\varepsilon s}{2}}(x-\varepsilon z/2, y)\mathbf{G}^*_{t+\frac{\varepsilon s}{2}}(x+\varepsilon z/2, y)\chi(y)dsdzdy,$$

we have, by an inverse Fourier transform,

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathbf{G}_{t-\frac{\varepsilon s}{2}}(x_0+\varepsilon\xi, y)\mathbf{G}^*_{t+\frac{\varepsilon s}{2}}(x_0+\varepsilon z, y)\chi(y)dy\\ &\qquad = \int_{\mathbb{R}^4} e^{-i(\omega s + k\cdot(z-\xi))}\mathbf{W}_\varepsilon(t,\omega,x_0+\frac{\varepsilon}{2}(z+\xi), k)d\omega dk.\end{aligned}$$

This allows us to recast $\mathbf{u}_R^\varepsilon$ as

$$\mathbf{u}_R^\varepsilon(\xi) \quad = \quad \int_{\mathbb{R}^3} \boldsymbol{\mathcal{J}}_\varepsilon(\xi,z)\mathbf{Q}(z)dz,$$

with

$$\boldsymbol{\mathcal{J}}_\varepsilon(\xi,z) = -\frac{\alpha}{2}\int_{\mathbb{R}^4} e^{ik\cdot(\xi-z)}\hat{\eta}(2\mu_\varepsilon\omega)\mathbf{W}_\varepsilon(1,\omega,x_0+\frac{\varepsilon}{2}(z+\xi),k)\Gamma A(x_0+\varepsilon z)d\omega dk.$$

We now pass to the limit adapting the results of [6] and [2]. In particular, the function f of [6] is a delta function here, and the function χ of [6] corresponds to square root of our function χ. We then find that the limit of $\mathbf{W}_\varepsilon$ is $\mathbf{W}+\mathbf{W}_1$, where $\mathbf{W}_1$ is not needed since $\mathbf{Q}$ is irrotational, and $\mathbf{W}$ reads

$$\mathbf{W}(t,\omega,x,k) = \frac{1}{(2\pi)^3}\sum_{\pm}\delta\left(\omega \mp \left(\frac{\bar{c}(x)}{c_-}\right)|k|\right) a_\pm(t,x,k)\,\mathbf{b}^\pm(x,k)\otimes\mathbf{b}^\pm(x,k),$$

where $a_\pm$ satisfy the transport equations stated in the theorem. We can then pass to the limit in $\boldsymbol{\mathcal{J}}_\varepsilon$ and find

$$\boldsymbol{\mathcal{J}}_\varepsilon(\xi,z)\mathbf{Q}(z) \to \boldsymbol{\mathcal{J}}(\xi-z)\mathbf{Q}(z) \qquad \text{as} \qquad \varepsilon\to 0,$$

where

$$\boldsymbol{\mathcal{J}}(v) = -\frac{\alpha}{2}\int_{\mathbb{R}^4} e^{ik\cdot v}\hat{\eta}(2\mu\omega)\mathbf{W}(1,\omega,x_0,k)\Gamma\mathbb{E}\{A(x_0)\}d\omega dk.$$

Integrating out the variable ω, and using (13) together with $\mathbf{b}^\pm\cdot\mathbb{E}\{A(x_0)\}\mathbf{b}^\pm = 1$, $\mathbf{b}^\pm\cdot\mathbb{E}\{A(x_0)\}\mathbf{b}^\mp = 0$, $\Gamma\mathbf{b}^\pm = \mathbf{b}^\mp$, finally yield the desired result.

A Modeling of time-dependent media

The main results of the paper are obtained thanks to the decomposition (8), which is derived for acoustic systems of the form (5). As we remarked in section 2, this imposes constraints on the structure of the time-dependent coefficients. This appendix revisits such constraints and considers more general time-dependent fluctuations whose effect can be computed in the simplified setting where $\chi(x) \equiv 1$.

We investigate three problems: in the first one, we obtain some necessary conditions on the time-dependent coefficients for systems of the form (1) to admit continuous in time solutions (more precisely for the solution vector to have some continuous components); we then consider the case of a time-independent impedance, for which we show that no refocused signal is generated and the effect of the time perturbation is a simple phase shift; finally, we study in a simple setting the coupling between the forward and time-reversed waves, and construct in particular discontinuous solutions when both the compressibility and density are singularly perturbed.

Many wave propagation problems with time-dependent coefficients may be modeled by an equation of the form

$$\partial_t \mathbf{u} = A_1(t,x) A_0(x,D)\mathbf{u}. \tag{19}$$

For the acoustic system in d dimensions, $\mathbf{u}(t,x) \in \mathbb{R}^m$ with $m = d+1$, and $A_0(x,D)$ (the symbol D stands for $-i\nabla$) is a (square) matrix-valued differential operator while $A_1(t,x)$ a square matrix. Accounting for perturbations both in the density and compressibility coefficients, denoted by V_ρ and V_κ, respectively, we have

$$\rho^{-1}(t,x) = \rho_0^{-1}(x)(1+V_\rho(t,x)), \qquad \kappa^{-1}(t,x) = \kappa_0^{-1}(x)(1+V_\kappa(t,x))$$

and

$$A_1(t,x) = \begin{pmatrix} (1+V_\rho(t,x))\mathbf{I}_d & \mathbf{0} \\ \mathbf{0}^T & 1+V_\kappa(t,x) \end{pmatrix}, \quad A_0(x,D) = \frac{1}{i}\begin{pmatrix} \mathbf{0}_d & \rho_0^{-1}(x)D \\ \kappa_0^{-1}(x)D^T & 0 \end{pmatrix}.$$

Above, $\mathbf{I}_d$ (resp $\mathbf{0}_d$) is the d-dimensional identity (resp. zero) matrix, and $\mathbf{0}$ is the zero column vector of dimension d.

Let us first consider the setting of section 2, with time-dependent fluctuations localized around a time T, leading in the limit $\tau \to 0$ to an equation of the form

$$\partial_t \mathbf{u} = A_0(x,D)\mathbf{u} + \delta_T(t) B(x,D)\mathbf{u}, \tag{20}$$

where δ_T is the Dirac function at T, $B = B_0 A_0$ and $A_1 = I + \delta_T(t) B_0$ for some matrix B_0. The above equation may be seen as the limit as $\tau \to 0$ of (5).

In order for the last term in the equation to be defined, we need Bu to be continuous at time $t = T$. Integrating the above equation between T^- and T^+ gives

$$u(T^+) - u(T^-) = 0 + (Bu)(T),$$

so that after applying the matrix operator B once more, we obtain that $B^2 u(T) = 0$. Since this needs to be valid for general vectors $u(T)$, we obtain that a quasi-necessary condition for the above equation to make sense is that $B^2 = 0$.

Constructing solutions to (20), at least formally, is central to our analysis in sections 3&4. Heuristically, the quality of refocusing and the possibility to properly define solutions to (20) are related as follows. We have seen in section 3 that a key ingredient for the error $\mathbf{R}^\tau$ to be small as $\tau \to 0$ is for $\nabla p(s) - \nabla p(s-\tau)$ to be small around the perturbation. In particular, $\mathbf{R}^\tau$ is of order τ when $\nabla \partial_t p$ is bounded independently of τ. The decomposition of the field $\mathbf{u}$ given in (8) can be generalized to the case of perturbations in both ρ and κ, with a new error term $\mathbf{R}^\tau$. The latter will be of order τ if all components of the (spatial derivatives) of the field admit uniformly bounded (in τ) time derivatives. Failure to construct solutions to (20) indicates a lack of uniform boundedness of the time derivatives of the field solution to the system with non-zero τ, which, as a consequence, yields a larger term $\mathbf{R}^\tau$ that is not of order τ and pollutes the refocused signal $\mathbf{u}_R$ in (8).

In the case of a system of acoustic equations, we observe, with $V_\rho(t,x) = V_\rho(t) = \alpha\delta_T(t)$ and $V_\kappa(t,x) = V_\kappa(t) = \beta\delta_T(t)$, that

$$B = B_0 A_0 \qquad \text{where} \qquad B_0 = \begin{pmatrix} \alpha \mathbf{I}_d & \mathbf{0} \\ \mathbf{0}^T & \beta \end{pmatrix},$$

and as a consequence

$$B(x,D)^2 = \alpha\beta A_0(x,D)^2.$$

Thus, in order for $B^2 = 0$ to hold, we need that either $\alpha = 0$ or $\beta = 0$. The analysis presented in this paper is based on the convenient form for $\mathbf{u}$ above. As we indicated in the text, it imposes fast variations of either ρ^{-1} or κ^{-1} but *not both* at the same time.

It turns out that other (and important) classes of rapid temporal fluctuations can be considered with some other quantities than $\mathbf{u}$ being discontinuous at time $t = T$. In the next paragraph, we consider spatially-independent fluctuations leaving the impedance constant. We show the physically expected result that no back-scattered signal is generated in such a situation. In the following paragraph, we generalize the analysis to still spatially-independent fluctuations that generate couplings between propagating modes and their time-reversed versions. Such an analysis is based on the existence of propagating modes and on fluctuations that generate couplings that can be computed explicitly. This rules out the important case of spatially-varying fluctuations (for instance $\chi(x)$ supported on a compact domain), whose effect on propagating modes is significantly more complex. This was the main reason for the introduction of (5).

The case of time-independent impedance. To motivate the analysis, let us consider the setting of constant coefficients (in space and time) and of plane wave solutions. We verify that

$$\mathbf{u}_\pm(t,x) = \mathbf{u}_\pm e^{i(c|k|t \mp k\cdot x)}$$

is a solution with $c = (\kappa\rho)^{-\frac{1}{2}}$ and

$$\mathbf{u}_\pm = \begin{pmatrix} \mp\zeta^{-1}\hat{k} \\ 1 \end{pmatrix}, \qquad \zeta = c\rho = \frac{1}{\kappa c} = \sqrt{\frac{\rho}{\kappa}}, \qquad \hat{k} = \frac{k}{|k|}.$$

Here, ζ is the impedance, the ratio of pressure over velocity in propagating plane waves. The reason for the presence of both signs $\pm$ is to present waves in pairs of forward- and backward- (time-reversed) propagating signals. The main influence of time-dependent fluctuations is to couple the two modes $u_\pm$ with each other and this is precisely how a time-reversed signal is generated. Rapid variations in ρ^{-1} and κ^{-1} can easily be analyzed within this framework of plane waves when the coefficients are spatially independent.

It is expected physically that any temporal fluctuations of κ and ρ such that ζ remains constant should not affect the dispersion relation and hence generate no coupling between the modes $\mathbf{u}_\pm(t,x)$. We now verify this in a slightly more general setting than the constant coefficients (in space) case. For the acoustic system, suppose that $V_\rho(t,x) = V_\rho(t) = V_\kappa(t)$, so that the impedance ζ does not depend on time. Then the matrix $A_1(t)$ in (19) is proportional to the identity matrix, and this leads to an equation of the form

$$\partial_t \mathbf{u} = a_1(t) A_0(x, D)\mathbf{u}, \tag{21}$$

with $a_1(t)$ a scalar, for instance constant (equal to 1) except in a small interval $(T - \tau/2, T + \tau/2)$, and A_0 is the time-independent system. Now, let us assume the existence of a (non-trivial) solution of the generalized eigenvalue problem

$$A_0(x, D)\mathbf{u}_0 = i\omega \mathbf{u}_0, \tag{22}$$

with ω real-valued since A_0 is hyperbolic. These are simply generalized plane waves. We then look for solutions of (21) of the form

$$\mathbf{u}(t, x) = q(t)\mathbf{u}_0(x)$$

with an equation for $q(t)$ then given by

$$\dot{q}(t) = i\omega a_1(t) q(t), \qquad \text{so that} \qquad q(t) = q(0) \exp\left\{ \int_0^t i\omega a_1(s) ds \right\}.$$

When a_1 is an approximation of $1 + \alpha\delta_T(t)$, we thus observe that the influence of the temporal fluctuations is to induce a phase shift given by $\alpha\omega$.

For acoustic waves with spatially invariant media, this corresponds to looking for a solution of the form

$$\mathbf{u}_+(t) = \begin{pmatrix} -\zeta^{-1}\hat{k} \\ 1 \end{pmatrix} q(t) e^{-ik\cdot x}$$

with $\dot{q}(t) = i|k|\zeta^{-1}\kappa^{-1}(t)q$ when ζ is *time-independent*, i.e., when $\rho(t)/\kappa(t) = \zeta^2$ is constant.

Note that in the above setting, $q(t)$ is not continuous at time $t = T$, so that we cannot write the equation $\dot{q}(t) = i\omega(1 + \alpha\delta_T(t))q(t)$ with a discontinuity in $\dot{q}$. Instead, it is $\frac{\dot{q}}{q}$ that is discontinuous, with a resulting phase shift. Hence, equation (20) is not satisfied as stated, but solutions can still be constructed provided the time fluctuations in the density and the compressibility are the same.

Finally, since (21) is linear, one can consider linear combinations of modes satisfying an equation of the form (22). They will remain uncoupled and each see an appropriate phase shift at $t = T$. We address the simplest of non-trivial mode couplings in the next paragraph.

Mode coupling with time-reversed mode. Consider the same equation for $\mathbf{u}$ as above with this time two modes (forward and time-reversed) given by

$$A_0(x, D)\mathbf{u}_\pm = \pm i\omega \mathbf{u}_\pm.$$

In this setting, we then look for solutions of (21) of the form

$$\mathbf{u}(t, x) = \mathbf{u}_+(x)q_+(t) + \mathbf{u}_-(x)q_-(t).$$

This would generalize in the setting with polarization (higher-dimensional eigenspaces as they appear in electromagnetism or elasticity) to $\mathbf{u}_\pm$ given by matrices of eigenvectors. In the acoustic case, write $\mathbf{u}_+(x) = (\mathbf{v}(x), w(x))^T$, for $\mathbf{v}$ a d-dimensional vector. It follows directly that $\mathbf{u}_- = (-\mathbf{v}(x), w(x))^T$. Denote then by $\mathcal{S}$ the span of $\mathbf{u}_+$ and $\mathbf{u}_-$. A short calculation shows that $A_1(t)\mathcal{S} \subset \mathcal{S}$, i.e., the temporal fluctuations acting on $\mathbf{u}_\pm$ do not generate anything that cannot be decomposed over these two modes, and more precisely

$$A_1(t)A_0(x,D)\mathbf{u}(t,x) = A_1(t)\left(\sum_{\pm} \pm i\omega q_\pm(t)\mathbf{u}_\pm(x)\right) = \sum_{\pm} \gamma_\pm(t)\mathbf{u}_\pm$$

where

$$\gamma_+(t) = i\omega\left(\Sigma(t)q_+(t) + \Delta(t)q_-(t)\right), \qquad \gamma_-(t) = i\omega\left(-\Sigma(t)q_-(t) - \Delta(t)q_+(t)\right)$$

and

$$\Sigma(t) = \frac{1}{2}\left(2 + V_\rho(t) + V_\kappa(t)\right), \qquad \Delta(t) = \frac{1}{2}\left(V_\rho(t) - V_\kappa(t)\right).$$

We thus obtain the differential equation

$$\mathbf{u}_+(x)\dot{q}_+(t) + \mathbf{u}_-(x)\dot{q}_-(t) = \mathbf{u}_+(x)\gamma_+(t) + \mathbf{u}_-(x)\gamma_-(t).$$

Writing $q = (q_+, q_-)^T$, we find, after appropriate projection of the above equation, the following system:

$$\dot{q}(t) = M(t)q(t), \qquad M(t) = i\omega\begin{pmatrix} \Sigma(t) & \Delta(t) \\ -\Delta(t) & -\Sigma(t) \end{pmatrix}. \tag{23}$$

The above system describes the coupling between the forward and time-reversed modes for any bounded perturbations such that $\rho(t,x)$ and $\kappa(t,x)$ are positive. There is a non-trivial coupling as soon as $\Delta(t) \neq 0$, that is when the fluctuations of the density and the compressibility are different. The case $\Delta(t) = 0$ was addressed in the preceding paragraph. Note that

$$\frac{d|q(t)|^2}{dt} = 4\Delta(t)\Im(q_-(t)q_+^*(t)),$$

and as a consequence the energy is conserved (only) when $\Delta(t) = 0$.

Assuming the perturbations are supported in $(T - \frac{\tau}{2}, T + \frac{\tau}{2})$ as in section 2, we can solve (23) on this interval and relate the initial and final amplitudes via a scattering matrix S_τ such that

$$q(T + \tau/2) = S_\tau q(T - \tau/2).$$

We can obtain simple expressions of S_τ in the following cases: when the perturbations are identical, when one is zero, or when they are constant in $(T - \tau/2, T + \tau/2)$. In the first case, we have, as already seen,

$$M(t) = i\omega\begin{pmatrix} 1 + V_\rho(t) & 0 \\ 0 & -(1 + V_\rho(t)) \end{pmatrix} \qquad \text{and then} \qquad S_\tau = \begin{pmatrix} s_\tau & 0 \\ 0 & s_\tau^* \end{pmatrix},$$

with $s_\tau = \exp(i\omega\int_{T-\frac{\tau}{2}}^{T+\frac{\tau}{2}}(1 + V_\rho(s))ds)$. When V_ρ is an approximation of a delta function at $t = T$ with weight α, we then find $S_\tau \to S_0$, where S_0 is diagonal with entries $s_0 = e^{i\omega\alpha}$ and s_0^*.

For the second case, we have, with for instance $V_\kappa = 0$,

$$M(t) = i\omega\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{V_\rho(t)}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right] := D + M_0(t).$$

Writing $q(t) = e^{tD}\mu(t)$, we find

$$\dot{\mu}(t) = V_\rho(t) M_1(t)\mu(t), \qquad M_1(t) = \frac{i\omega}{2}\begin{pmatrix} 1 & m^*(t) \\ -m(t) & -1 \end{pmatrix}, \qquad m(t) = e^{2i\omega t}.$$

Interestingly, while the above equation does not seem to be exactly solvable in general, it is when $V_\rho(t) = \alpha\delta_T(t)$ since we can exploit the fact that $(M_1(T))^2 = 0$. We then find

$$\mu(T^+) = (\mathbf{I}_2 + \alpha M_1(T))\mu(T^-),$$

and the asymptotic scattering matrix is

$$S_0 = \mathbf{I}_2 + \alpha e^{TD} M_1(T) e^{-TD} = \mathbf{I}_2 + \frac{i\alpha\omega}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}. \tag{24}$$

Note that, while $q(t)$ is discontinuous at $t = T$, it is possible to construct solutions to (23) as $\tau \to 0$. These solutions do not satisfy (23) quite as stated but rather in a proper weaker sense. To make the connection with the first paragraph of the appendix, note that the relevant quantities to address the solvability of (20) are $q_+(t) \pm q_-(t)$ since

$$\mathbf{u}(t,x) = \begin{pmatrix} (q_+(t) - q_-(t))\mathbf{v}(x) \\ (q_+(t) + q_-(t))w(x) \end{pmatrix}.$$

We then deduce from the scattering matrix (24) that $q_+(t) + q_-(t)$ is continuous at T, and therefore that the last component of $\mathbf{u}$ is continuous, which is enough to make sense of (20) when $V_\kappa = 0$.

When both perturbations are constant on $(T - \tau/2, T + \tau/2)$ and of the form $V_\rho(t) = \alpha\eta((t-T)/\tau)$ and $V_\rho(t) = \beta\eta((t-T)/\tau)$, where η is defined in section 2, (23) can be solved exactly in $(T - \tau/2, T + \tau/2)$, and we find, with $\alpha_0 = 1 + \alpha$, $\beta_0 = 1 + \beta$, $\theta = \omega\tau\sqrt{\alpha_0\beta_0}$, $a = \sqrt{\alpha_0/\beta_0}$,

$$S_\tau = 4\begin{pmatrix} s_1 & s_2^* \\ s_2 & s_1^* \end{pmatrix}, \quad s_1 = \cos\theta + \frac{i}{2}\left(\frac{1}{a} + a\right)\sin\theta, \quad s_2 = \frac{i}{2}\left(\frac{1}{a} - a\right)\sin\theta.$$

The above expressions show in particular that the forward and the time-reversed modes cannot be exchanged since s_1 cannot vanish (since a is positive). The largest transfer (largest ratio $|s_2|/|s_1|$) is obtained for $\theta = \frac{\pi}{2}$ and a either very large or very small, i.e., when only one of the coefficients varies.

Note that we can address the limit $\tau \to 0$ as follows: rescaling α and β as α/τ and β/τ, we can take the limit of S_τ and find the same expression as above with now $\theta = \omega\sqrt{\alpha\beta}$ and $a = \sqrt{\alpha/\beta}$. Again, q is discontinuous at T but still satisfies (23) in a proper sense as $\tau \to 0$.

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