Uniqueness for Weak Solutions to Navier Stokes

We proved the following existence result for Navier-Stokes:

Theorem (Existence of a weak solution, $n \leq 4$) For every $\vec{f} \in L^2[0,T : V']$ and each $u_0 \in H$ there exists at least one $\vec{u} \in L^2[0,T : V]$ that is a weak solution of the N-S system. In addition, this weak solution satisfies:

(a) $\vec{u} \in L^\infty[0,T : H]$ *and* $\vec{u}' \in L^{4/n}[0,T : V']$

(b) $\vec{u}(t)$ is weakly continuous in H;

Uniqueness in the Case n $=$ 2

When n=2, the weak solution has additional regularity and is, in fact, unique.

Theorem (Uniqueness of the weak solution, $n = 2$) In the case n=2, the weak solution of the Navier Stokes equations satisfies:

i) $\vec{u} \in L^2[0,T : V]$ *and* $\vec{u}' \in L^2[0,T : V']$ hence $\vec{u} \sim \tilde{u}(t) \in C[0,T : H]$

ii) $\vec{u}(t)$ is unique

Proof- We begin by repeating an estimate for the nonlinear term in the N-S system. For arbitrary $\vec{u}, \vec{v}, \vec{w} \in H_0^1(U)^n$, we have (by the extended Holder ineqality followed by the C-S inequality)

$$|b(\vec{u},\vec{v},\vec{w})| \leq \sum_{i,j=1}^2 \int_U |u_i\, \partial_i v_j\, w_j|\, dx \leq \sum_{i,j=1}^2 \|u_i\|_4 \|\partial_i v_j\|_2 \|w_j\|_4$$

$$\leq \left(\sum_{i,j=1}^2 \|\partial_i v_j\|_2^2\right)^{1/2} \left(\sum_{i=1}^2 \|u_i\|_4^2\right)^{1/2} \left(\sum_{j=1}^2 \|w_j\|_4^2\right)^{1/2}$$

In addition, the Sobolev inequality implies

$$\|u\|_q \leq C \|\nabla u\|_p^\lambda \|u\|_r^{1-\lambda} \qquad \forall u \in C_0^\infty(U)$$

where

$$0 \leq \lambda \leq 1, \qquad 1/q = \lambda(1/p - 1/n) + (1-\lambda)\,1/r$$

If we choose q=4 and p=r=2 so that $\lambda = n/4$, then with $n = 2$, we have

$$\|u\|_4 \leq C \|\nabla u\|_2^{1/2} \|u\|_2^{1/2} \qquad \forall u \in H_0^1(U)$$

and, combined with the previous estimate,

$$\left(\sum_{i=1}^{2} \|u_i\|_4^2\right) \leq C\sum_{i=1}^{2} \|u_i\|_2 \, \|\nabla u_i\|_2 \leq C \, \|\vec{u}\|_H \, \|\vec{u}\|_V.$$

This leads to the following estimate on $|b(\vec{u},\vec{v},\vec{w})|$

$$|b(\vec{u},\vec{v},\vec{w})| \leq C \, \left(\|\vec{u}\|_H \, \|\vec{u}\|_V\right)^{1/2}.\|v\|_V \, \left(\|\vec{w}\|_H \, \|\vec{w}\|_V.\right)^{1/2}$$

and

$$|b(\vec{u},\vec{u},\vec{v})| = |-b(\vec{u},\vec{v},\vec{u})| \leq C \, \|\vec{u}\|_H \, \|\vec{u}\|_V.\|v\|_V \qquad \forall \vec{u},\vec{v} \in V$$

This implies that

$$\|B(\vec{u}(t))\|_{V'} \leq C \, \|\vec{u}\|_H \, \|\vec{u}\|_V$$

Since the weak solution, $\vec{u}(t)$, belongs to $L^{\infty}[0,T : H] \cap L^2[0,T : V]$, we have

$$\int_0^T \|B(\vec{u}(t))\|_{V'}^2 dt \leq C\int_0^T \left(\|\vec{u}(t)\|_H \, \|\vec{u}(t)\|_V\right)^2 dt \leq C\|\vec{u}\|_{\infty}^2 \, \|\vec{u}\|_{L^2[0,T:V]}^2$$

hence $\|B(\vec{u}(\bullet))\|_{V'} \leq C\|\vec{u}\|_{\infty} \, \|\vec{u}\|_{L^2[0,T:V]}$ and $B(\vec{u}(t)) \in L^2[0,T : V']$
Then $\vec{u}'(t)$ also belongs to $L^2[0,T : V']$ and then it follows from Theorem 3, p287 in Evans that

$$\vec{u} \in L^2[0,T : V] \quad and \quad \vec{u}' \in L^2[0,T : V'] \quad \text{implies} \quad \vec{u}(t) \sim \tilde{u}(t) \in C[0,T : H]$$

Now, for $\vec{u} \in L^2[0,T : V]$ *and* $\vec{u}' \in L^2[0,T : V']$ we have

$$\langle \vec{u}'(t), \vec{u}(t)\rangle_{V\times V'} = \tfrac{1}{2}\tfrac{d}{dt}\|\vec{u}(t)\|_H^2$$

Then, if $\vec{u}_1(t)$, $\vec{u}_2(t)$ are two weak solutions for N-S, let $\vec{u}(t) = \vec{u}_1(t) - \vec{u}_2(t)$. Then

$$\vec{u}'(t) + \upsilon A\vec{u}(t) = B[\vec{u}_2(t)] - B[\vec{u}_1(t)], \qquad \vec{u}(0) = 0,$$

and $$\tfrac{d}{dt}\|\vec{u}(t)\|_H^2 + 2\upsilon\|\vec{u}(t)\|_V^2 = 2b(\vec{u}_2(t),\vec{u}_2(t),\vec{u}(t)) - 2b(\vec{u}_1(t),\vec{u}_1(t),\vec{u}(t)).$$

But $$b(\vec{u}_2(t),\vec{u}_2(t),\vec{u}(t)) - b(\vec{u}_1(t),\vec{u}_1(t),\vec{u}(t)) = -[b(\vec{u}_1(t),\vec{u}_1(t),\vec{u}(t)) - b(\vec{u}_2(t),\vec{u}_2(t),\vec{u}(t))]$$

$$= -[b(\vec{u}_1(t),\vec{u}_1(t),\vec{u}(t)) - b(\vec{u}_2(t),\vec{u}_1(t),\vec{u}(t)) + b(\vec{u}_2(t),\vec{u}_1(t),\vec{u}(t)) - b(\vec{u}_2(t),\vec{u}_2(t),\vec{u}(t))]$$

and $$b(\vec{u}_1(t),\vec{u}_1(t),\vec{u}(t)) - b(\vec{u}_2(t),\vec{u}_1(t),\vec{u}(t)) = b(\vec{u}(t),\vec{u}_1(t),\vec{u}(t))$$

$$b(\vec{u}_2(t),\vec{u}_1(t),\vec{u}(t)) - b(\vec{u}_2(t),\vec{u}_2(t),\vec{u}(t)) = b(\vec{u}(t),\vec{u}(t),\vec{u}(t)) = 0$$

Then $$\tfrac{d}{dt}\|\vec{u}(t)\|_H^2 + 2\upsilon\|\vec{u}(t)\|_V^2 = -2b(\vec{u}(t),\vec{u}_1(t),\vec{u}(t)).$$

But as we showed previously, (and using $abc \leq \frac{\upsilon}{C}a^2 + \frac{C}{\upsilon}b^2c^2$)

$$|b(\vec{u}(t),\vec{u}_1(t),\vec{u}(t))| \leq C\|\vec{u}(t)\|_H\|\vec{u}(t)\|_V\|\vec{u}_1(t)\|_V$$
$$\leq C[\frac{\upsilon}{C}\|\vec{u}(t)\|_V^2 + \frac{C}{\upsilon}\|\vec{u}(t)\|_H^2\|\vec{u}_1(t)\|_V^2]$$

and therefore,

$$\frac{d}{dt}\|\vec{u}(t)\|_H^2 \leq \frac{2C^2}{\upsilon}\|\vec{u}(t)\|_H^2\|\vec{u}_1(t)\|_V^2$$

Since $\|\vec{u}_1(t)\|_V^2 \in L^1[0,T]$, we get

$$\frac{d}{dt}\left(\|\vec{u}(t)\|_H^2 \exp\left\{-\frac{2C^2}{\upsilon}\int_0^t\|\vec{u}_1(s)\|_V^2 ds\right\}\right) \leq 0$$

i.e., $\|\vec{u}(t)\|_H^2 \leq \|\vec{u}(0)\|_H^2 = 0$, or $\vec{u}_1(t) = \vec{u}_2(t)$.

Summarizing,

> when n = 2 the NS system has a unique solution $\vec{u}(t) \in C[0,T:H] \cap L^2[0,T:V]$, with $\vec{u}'(t) \in L^2[0,T:V']$. In addition, $\vec{u}(t) \in L^4[U_T]$

Since the solution is unique in the case $n = 2$ and is not unique in the case $n \geq 3$, it must be that the assumptions under which a physical flow can be treated as a 2-dimensional flow are also the conditions that preclude turbulence and the consequent loss of uniqueness. Thus a flow which can be treated as 2-dimensional under some conditions (e.g., shallow flow with low flow velocity) may under other conditions behave in ways that cannot be viewed as 2-dimensional, even though the flow may still be a shallow flow.

Uniqueness in the Case n = 3

Recall the corollary to the Sobolev inequality implies

$$\|u\|_4 \leq C\|\nabla u\|_2^{n/4}\|u\|_2^{1-n/4} \qquad \forall u \in H_0^1(U)$$

so that, in the case n = 3, we have

$$\|u\|_4 \leq C\|\nabla u\|_2^{3/4}\|u\|_2^{1/4} \qquad \forall u \in H_0^1(U)$$

This leads to the result that the weak solution of N-S satisfies,

$$\vec{u}(t) \in L^{8/3}[0,T:L^4[U]^n], \qquad and \qquad \vec{u}'(t) \in L^{4/3}[0,T:V']$$

Note that this is strictly weaker than what was true of the solution in the case n = 2.

Theorem Suppose n = 3 and the weak solution of the N-S system satisfies:

$$\vec{u}(t) \in L^\infty[0,T : H] \cap L^2[0,T : V] \quad and \quad \vec{u}(t) \in L^8[0,T : L^4[U]^n],$$

Then, under these conditions, $\vec{u}(t)$ is unique and $\vec{u}(t) \in C[0,T : H]$

Proof- Supppose $\vec{u}(t)$ is a weak solution for N-S with the additional regularity indicated above.
Then

$$|b(\vec{u}(t),\vec{u}(t),\vec{v})| \leq C\|\vec{u}(t)\|_4^2\|\vec{v}\|_V \qquad \forall\, \vec{u}(t),\vec{v} \in V,$$

implies

$$\|B(\vec{u}(t))\|_{V'} \leq C\|\vec{u}(t)\|_4^2$$

and
$$\int_0^T \|B(\vec{u}(t))\|_{V'}^p\, dt \leq C\int_0^T \|\vec{u}(t)\|_4^{2p}dt \leq C\int_0^T \|\vec{u}(t)\|_V^{np/2}dt$$

It follows that $B(\vec{u}(t)) \in L^p[0,T : V']$ if $\vec{u}(t) \in L^2[0,T : V]$ $and \;\; np/2 \leq 2; i.e.,$

$$B(\vec{u}(t)) \in L^{4/n}[0,T : V'] = L^{4/3}[0,T : V'].$$

On the other hand, if $\vec{u}(t) \in L^8[0,T : L^4[U]^n]$, then

$$\int_0^T \|\vec{u}(t)\|_4^{2p}dt < \infty \quad for \;\; p \leq 4$$

and

$$B(\vec{u}(t)) \in L^2[0,T : V'] \quad \text{(at least)}$$

Then it follows from the N-S equation that $\vec{u}'(t) \in L^2[0,T : V']$ and this, together with $\vec{u}(t) \in L^2[0,T : V]$ imply $\vec{u}(t) \in C[0,T : H]$.

Now observe that

$$|b(\vec{u}(t),\vec{u}(t),\vec{v})| \leq C_0\|\vec{u}(t)\|_4\|\vec{u}(t)\|_V\|\vec{v}\|_4$$

$$\leq C_1\left(\|\vec{u}(t)\|_2^{1/4}\|\vec{u}(t)\|_V^{3/4}\right)\|\vec{u}(t)\|_V\|\vec{v}\|_4$$

$$\leq C_1\|\vec{u}(t)\|_H^{1/4}\|\vec{u}(t)\|_V^{7/4}\|\vec{v}\|_4$$

Now suppose $\vec{u}_1(t)$, $\vec{u}_2(t)$ are two weak solutions for N-S both of which have the additional regularity of the hypotheses. Let $\vec{u}(t) = \vec{u}_1(t) - \vec{u}_2(t)$, and note that as in the n = 2 proof

$$\frac{d}{dt}\|\vec{u}(t)\|_H^2 + 2\upsilon\|\vec{u}(t)\|_V^2 = 2b(\vec{u}(t),\vec{u}(t),\vec{u}_2(t)).$$

Also

$$|b(\vec{u}(t),\vec{u}(t),\vec{u}_2(t)).| \leq C_1\|\vec{u}(t)\|_H^{1/4}\|\vec{u}(t)\|_V^{7/4}\|\vec{u}_2(t)\|_4 \leq \upsilon\|\vec{u}(t)\|_V^2 + C_2\|\vec{u}(t)\|_H^2\|\vec{u}_2(t)\|_4^8$$

and so $\frac{d}{dt}\|\vec{u}(t)\|_H^2 \leq C_2\|\vec{u}_2(t)\|_4^8\,\|\vec{u}(t)\|_H^2.$

Since $\|\vec{u}_2(t)\|_4^8 \in L^1[0,T],$ we can finish the proof as in the n=2 case.

In the step,

$$\|\vec{u}(t)\|_H^{1/4}\|\vec{u}(t)\|_V^{7/4}\|\vec{u}_2(t)\|_4 \leq \upsilon\|\vec{u}(t)\|_V^2 + C_2\|\vec{u}(t)\|_H^2\|\vec{u}_2(t)\|_4^8$$

we used the following version of Young's inequality,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \epsilon a^p + C(\epsilon)b^q, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad a,b > 0$$