The Stokes Operator

A Summary of the Function Spaces

Suppose $U \subset R^n$ is open, bounded and has a smooth boundary ∂U. Then we have defined the following spaces of functions

$$E(U) = \{\vec{u} \in L^2(U)^n : div\,\vec{u} \in L^2(U)\}.$$

Then $E(U)$ is a Hilbert space and supports a trace operator

$$T : E(U) \rightarrow H^{-1/2}(\partial U) \quad (onto)$$

for which the following integration by parts formula then holds

$$((\vec{u}, \nabla w))_0 + (div\,\vec{u}, w)_0 = \langle T\vec{u}, T_0 w\rangle \qquad \forall \vec{u} \in E(U),\ w \in H^1(U).$$

The fact that the divergence and gradient operators are transposes of one another implies that the annihilator of the subspace

$$K = \{\vec{\phi} \in D(U)^n : div\,\vec{\phi} = 0\}$$

is composed of gradients. A typical result asserts that

$$\vec{f} \in K^0 \cap H^{-1}(U)^n \quad \Leftrightarrow \quad \vec{f} = \nabla p \text{ for some } p \in L^2(U).$$

If we define $H =$ the completion of K in the norm of $L^2(U)^n$
$V =$ the completion of K in the norm of $H^1(U)^n$,

then

$$H = \left\{\vec{u} \in L^2(U)^n : div\,\vec{u} = 0,\ T\vec{u} = 0\right\}$$
$$V = \{\vec{u} \in H_0^1(U)^n : div\,\vec{u} = 0\},$$

and

$$L^2(U)^n = H \oplus H^\perp = H \oplus H_1 \oplus H_2$$

where

$$H_1 = \left\{\vec{u} \in L^2(U)^n : \vec{u} = \nabla p,\ p \in H^1(U),\ \nabla^2 p = 0\right\}$$
$$H_2 = \left\{\vec{u} \in L^2(U)^n : \vec{u} = \nabla p,\ p \in H_0^1(U)\right\}.$$

Then

$$K \subset V \subset H \subset E(U) \subset L^2(U)^n \subset H^{-1}(U)^n \subset D'(U)^n$$

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The Hilbert space projection theorem implies the existence of projections

$$P : L^2(U)^n \to H \qquad and \quad Q : L^2(U)^n \to H^\perp$$

where

$$P\vec{u} = \vec{u}_0 = \vec{u} - \vec{u}_1 - \vec{u}_2.$$

In addition, P maps $H_0^1(U)^n$ continuously into $H^1(U)^n$. To see this, suppose $\vec{u} \in H_0^1(U)^n$. Then

$$\nabla^2 q = div\,\vec{u} \in L^2(U)$$
$$q \in H_0^1(U),$$

has a unique solution $q \in H^2(U),\ and\ \vec{u}_2 = \nabla q \in H^1(U)^n$.
In addition,

$$\vec{u} - \vec{u}_2 \in H^1(U)^n,\ so \quad T(\vec{u} - \vec{u}_2) \in H^{1/2}(\partial U)$$

and

$$\nabla^2 p = 0 \qquad in\ U$$
$$\partial_N p = T(\vec{u} - \vec{u}_2),$$

has a solution $p \in H^2(U),\ and\ \vec{u}_1 = \nabla p \in H^1(U)^n$. Then

$$\vec{u}_0 = \vec{u} - \vec{u}_1 - \vec{u}_2 \in H^1(U)^n$$

and

$$P : H_0^1(U)^n \to H^1(U)^n$$

is bounded since

$$\|P\vec{u}\|^2_{H^1(U)^n} = \|\vec{u}_0\|^2_{H^1(U)^n} = \|\vec{u}\|^2_{H^1(U)^n} - \|\vec{u}_1 + \vec{u}_2\|^2_{H^1(U)^n} \le \|\vec{u}\|_{H^1(U)^n}$$

The Stokes Operator

To say $\vec{u} \in V$ satisfies $\quad -\nu\nabla^2\vec{u} + \nabla p = \vec{f} \qquad$ for $\vec{f} \in L^2(U)^n$

means $\qquad \nu[[\vec{u},\vec{v}]] = ((\vec{f},\vec{v}))_0 \quad$ for all $\vec{v} \in V$.

More precisely, since $\nabla^2\vec{u} \in L^2(U)^n,\quad \nabla^2\vec{u} = \vec{w} + \vec{z},\ for\ \ \vec{w} \in H\ and\ \vec{z} \in H^\perp; i.e.$

$$((\nabla^2\vec{u},\vec{v}))_0 = ((\vec{w},\vec{v}))_0 + ((\vec{z},\vec{v}))_0 = ((\vec{w},\vec{v}))_0,$$

since

$$((\vec{z},\vec{v}))_0 = ((\nabla q,\vec{v}))_0 = \langle T\vec{v}, T_0 q\rangle - (div\,\vec{v}, q)_0 = 0 \quad \forall\vec{v} \in V.$$

Of course, $\qquad ((\nabla p,\vec{v}))_0 = 0, \quad$ and thus

$$((-\nu\nabla^2\vec{u} + \nabla p - \vec{f},\vec{v}))_0 = \ ((-\nu P\nabla^2\vec{u} - P\vec{f},\vec{v}))_0 = 0 \qquad \forall\vec{v} \in V$$

This means $A\vec{u} = P\vec{f},$ where $A\vec{u} = -\nu P\nabla^2\vec{u}$ *and* $P : L^2(U)^n \to H$ denotes the projection into H. Note that P does not commute with $-\nabla^2$, in general. Now let

$$D_A = \{\vec{u} \in V : A\vec{u} \in H\} = \{\vec{u} \in H^2(U)^n \cap V\}$$

$$= \left\{\vec{u} \in H^2(U)^n \cap H_0^1(U)^n : div\,\vec{u} = 0\right\}$$

Then we can show that A is self adjoint as an operator from D_A to H.

Lemma 1- A is symmetric; i.e., $((A\vec{u},\vec{v}))_0 = ((\vec{u},A\vec{v}))_0 \quad \forall\vec{u},\vec{v} \in D_A$

Proof- For $\vec{\phi},\vec{\psi} \in K = \{$ the divergence free test functions$\}$, $P\vec{\phi} = \vec{\phi},$ and $P\vec{\psi} = \vec{\psi},$ so

$$((A\vec{\phi},\vec{\psi}))_0 = ((-\nu\nabla^2\vec{\phi},\vec{\psi}))_0 = \nu[[\vec{\phi},\vec{\psi}]] = \nu[[\vec{\psi},\vec{\phi}]] = ((A\vec{\psi},\vec{\phi}))_0$$

Now K is dense in V and hence then also dense in D_A and so we can extend the equality here to the following cases

$$((A\vec{u},\vec{v}))_0 = [[\vec{u},\vec{v}]] \quad \text{for } \vec{u} \in D_A, \quad \vec{v} \in V$$

$$((\vec{u},A\vec{v}))_0 = [[\vec{u},\vec{v}]] \quad \text{for } \vec{v} \in D_A, \quad \vec{u} \in V$$

It follows from this that $((A\vec{u},\vec{v}))_0 = ((\vec{u},A\vec{v}))_0 \quad \forall\vec{u},\vec{v} \in D_A$ ■

Lemma 2- A is self adjoint

Proof- Fix $\vec{u} \in H,$ and define $F(\vec{v}) = ((\vec{u},A\vec{v}))_0$ for $\vec{v} \in D_A.$ Then F is clearly linear but is not necessarily bounded. However, let D_{A^*} denote the set of all $\vec{u}$ in H such that F is bounded. It is evident that zero belongs to D_{A^*} and it follows from lemma 1 that $D_A \subset D_{A^*}$;

i.e., $\quad F(\vec{v}) = ((\vec{u},A\vec{v}))_0 = ((A\vec{u},\vec{v}))_0$ if both $\vec{u},\vec{v} \in D_A.$

Then $\vec{v}_n \to \vec{v},$ in H implies $F(\vec{v}_n) = ((A\vec{u},\vec{v}_n))_0 \to ((A\vec{u},\vec{v}))_0$ in R, so F is continuous. Now it follows from the Riesz theorem that for each $\vec{u} \in D_{A^*}$ there is a unique $\vec{f}_u$ in H such that $F(\vec{v}) = ((\vec{f}_u,\vec{v}))_0$ for $\vec{v} \in D_A$. Note that since D_A is dense in H, $\vec{f}_u \in H$ is uniquely determined by $\vec{u} \in D_{A^*}$. If we denote this correspondence between $\vec{u}$ and $\vec{f}_u$ by writing $\vec{f}_u = A^*\vec{u}$ for all $\vec{u} \in D_{A^*}$, then

$$\forall\vec{u} \in D_{A^*} \quad ((\vec{u},A\vec{v}))_0 = ((A^*\vec{u},\vec{v}))_0 \quad \forall\,\vec{v} \in D_A.$$

For any $\vec{u} \in D_{A^*}$ we have $A^*\vec{u} \in H$, so we can use the existence theorem for the Stokes problem to find a unique $\vec{w} \in D_{A^*}$ such that

$$\begin{aligned} -\nu\nabla^2\vec{w} + \nabla p &= A^*\vec{u} && in\ U \\ div\ \vec{w} &= 0 && in\ U \\ \vec{w} &= 0 && on\ \Gamma \end{aligned}$$

i.e.,

$\vec{w} \in D_{A^*}$ is such that $A\vec{w} = A^*\vec{u} \in H$

Now we claim that $\vec{w} = \vec{u}$. To see this, consider $((\vec{z},\vec{u} - \vec{w}))_0$ for $\vec{z} \in H$, arbitrary. Using the existence theorem for the Stokes problem once again, we find $\vec{v} \in D_A$ such that $A\vec{v} = \vec{z}$. Then

$$((\vec{z},\vec{u} - \vec{w}))_0 = ((\vec{z},\vec{u}))_0 - ((\vec{z},\vec{w}))_0 = ((A\vec{v},\vec{u}))_0 - ((A\vec{v},\vec{w}))_0$$

But $((A\vec{v},\vec{u}))_0 = ((\vec{v},A^*\vec{u}))_0$ since $\vec{u} \in D_{A^*}$

and $((A\vec{v},\vec{w}))_0 = ((\vec{v},A\vec{w}))_0$ since $\vec{w},\vec{v} \in D_A$.

Then

$$((\vec{z},\vec{u} - \vec{w}))_0 = ((\vec{v},A^*\vec{u}))_0 - ((\vec{v},A\vec{w}))_0 = 0 \quad \forall\vec{z} \in H$$

i.e., for every $\vec{u} \in D_{A^*}$, $\vec{u} = \vec{w} \in D_A$ and $A\vec{u} = A^*\vec{u}$, so A is self adjoint.■

Theorem 1- The inverse of $A : D_A \to H$ is compact

Proof- Since $A\vec{u} = \vec{f}$ is uniquely solvable for every $\vec{f} \in H$, the inverse of A exists. From the regularity results for the Stokes problem, we have

$$||\vec{u}||_{H^2} + ||p||_{H^1} \leq C||f||_{L^2} \text{ i.e.,} \quad ||A^{-1}f||_{H^2} \leq C||f||_{L^2}.$$

which implies $A^{-1} : H \to D_A = H^2(U)^n \cap V$ is bounded. But the inclusion, $V \hookrightarrow H$ is compact (by Rellich's lemma) hence $A^{-1} : H \to H$ is compact.■

Note that $K = A^{-1}$ is compact and self-adjoint since A is self adjoint. Then K has an orthonormal (in H) family of eigenfunctions $\{\vec{w}_j\}$ that is complete in H (and in V). Then

$$A\vec{w}_j = \lambda_j\vec{w}_j, \text{ where } 0 < \lambda_1 \leq \lambda_2 \leq \bullet\bullet\bullet \to \infty$$

It can be shown that if $\partial U \in C^m$, then $\vec{w}_j \in H^m(U)^n$, $m \geq 2$.

Interpolation Spaces

We can use the orthonormal (in H) family of eigenfunctions $\{\vec{w}_j\}$ for K to build a sequence of Hilbert spaces all contained in H. For $\vec{u} \in H$,

$$\vec{u} = \sum_{j\geq 1}((\vec{u},\vec{w}_j))_0\ \vec{w}_j = \sum_{j\geq 1} U_j\ \vec{w}_j$$

and

$$\|\vec{u}\|_H^2 = \sum_{j\geq 1}|((\vec{u},\vec{w}_j))_0|^2 = \sum_{j\geq 1}|\ U_j|^2$$

For $u \in D_A$ $$A\ \vec{u} = \sum_{j\geq 1}((\vec{u},\vec{w}_j))_0\ A\vec{w}_j = \sum_{j\geq 1} U_j\ \lambda_j\vec{w}_j$$

and $$\|A\vec{u}\|_H^2 = \sum_{j\geq 1}\lambda_j^2|\ U_j\ |^2.$$

For $\alpha > 0$, define $$H_\alpha = D(A^\alpha) = \left\{\vec{u} \in H : \sum_{j\geq 1}\lambda_j^{2\alpha}|U_j|^2 < \infty\right\}$$

and

$$\forall\vec{u} \in H_\alpha \qquad A^\alpha\vec{u} = \sum_{j\geq 1}((\vec{u},\vec{w}_j))_0\ A^\alpha\vec{w}_j = \sum_{j\geq 1} U_j\ \lambda_j^\alpha\vec{w}_j.$$

Evidently, $$\|\vec{u}\|_\alpha^2 = \|\vec{u}\|_H^2 + \|A^\alpha\vec{u}\|_H^2$$

is the graph norm on D_A, and thus it follows from the closed graph theorem that H_α is a Banach space for this norm. Of course the norm then supports the inner product

$$(\vec{u},\vec{v})_\alpha = (\vec{u},\vec{v})_H + (A^{\alpha/2}\vec{u},A^{\alpha/2}\vec{v})_H = \sum_{j\geq 1}(1+|\lambda_j|^{2\alpha})|U_jV_j|$$

and H_α becomes a Hilbert space for this inner product. In addition, since $\lambda_j > 0$ for all j, this inner product is equivalent to

$$((\vec{u},\vec{v}))_\alpha = (A^{\alpha/2}\vec{u},A^{\alpha/2}\vec{v})_H = \sum_{j\geq 1}\lambda_j^{2\alpha}|U_jV_j|.$$

It is clear from the definitions that $H_0 = H$ and $H_1 = D_A$, and in addition, we can show that $H_{1/2} = V$. To see this, note that for $\vec{u} \in V$,

$$\|\vec{u}\|_V^2 = [[\vec{u},\vec{u}]] = ((A\vec{u},\vec{u}))_0 = \left(\sum_{j\geq 1} U_j\lambda_j\vec{w}_j, \sum_{k\geq 1} U_k\vec{w}_k\right)_0$$

$$= \sum_{j\geq 1}\sum_{k\geq 1} U_j\ \lambda_j U_k((\vec{w}_j,\vec{w}_k))_0 = \sum_{j\geq 1}|U_j|^2\ \lambda_j = ((\vec{u},\vec{u}))_{1/2} = \|\vec{u}\|_{1/2}^2$$

The spaces H_α are a continuously distributed scale of Hilbert spaces for $0 \leq \alpha \leq 1$; i.e.,

$$H_0^1(U) \cap H^2(U) \subset D_A = H_1 \subset H_\alpha \subset H_0 = H^0(U), \qquad 0 \leq \alpha \leq 1.$$

We can use arguments similar to those used in the development of the spaces $H^s(R^n)$ to show that

H_α is continuously embedded in $W^{p,q}(U)^n$ if $\left\{ \begin{array}{c} 2\alpha > p \\ 2\alpha - n/2 > p - n/q \end{array} \right\}$

H_α is continuously embedded in $C^m(\bar{U})^n$ if $2\alpha - n/2 > m$

In addition, we have the following interpolation results:

Lemma 3- For $a \leq \alpha \leq b$, and $\vec{u} \in H_b$

$$||\vec{u}||_\alpha \leq ||\vec{u}||_b^{1-\theta}\, ||\vec{u}||_a^\theta \qquad \theta = \tfrac{b-\alpha}{b-a}$$

Proof- For $\alpha \leq b$, and $\vec{u} \in H_b$, it follows that $\vec{u} \in H_\alpha$ and

$$||\vec{u}||_\alpha^2 = \sum_{j\geq 1} \lambda_j^{2\alpha}|U_j|^2 = \sum_{j\geq 1}(\lambda_j^{2b(1-\theta)}|U_j|^{2(1-\theta)})\left(\lambda_j^{2a\theta}|U_j|^{2\theta}\right)$$

Now use the Hölder inequality with $p = 1/(1-\theta)$ and $q = 1/\theta$, to get

$$||\vec{u}||_\alpha^2 = \left(\sum_{j\geq 1} \lambda_j^{2b}|U_j|^2\right)^{1-\theta}\left(\sum_{j\geq 1} \lambda_j^{2a}|U_j|^2\right)^\theta$$

from which it follows that $||\vec{u}||_\alpha \leq ||\vec{u}||_b^{1-\theta}||\vec{u}||_a^\theta$.■

The particular choice a=0, b=1 leads to

$$||\vec{u}||_\alpha \leq ||\vec{u}||_1^{1-\theta}||\vec{u}||_0^\theta = ||A\vec{u}||_H^{1-\theta}||\vec{u}||_H^\theta \qquad \forall\vec{u} \in D_A$$