The Stokes System

Settings for the Stokes Problem

As a first step to considering the Navier-Stokes equations, we will consider various settings for the Stokes problem of finding $\vec{u}, p$ which satisfy

$$\begin{aligned} -\nu\nabla^2\vec{u} + \nabla p &= \vec{f} \qquad in\ U \subset R^n \qquad\qquad (1) \\ div\ \vec{u} &= 0. \end{aligned}$$

We could consider this system of equations on:

1. **a**. $U = R^n$: in this case we would have to impose some "behavior at ∞" conditions such as $\vec{u} \to \vec{\psi}\ as\ |\vec{x}| \to \infty$

b. $U \subset R^n$, open bounded with smooth boundary ∂U. In this case we would impose conditions on ∂U such as $\vec{u} = \vec{0}\ on\ \partial U$. The more smoothness we require of ∂U the easier the proofs become but the less realistic the problem becomes.

c. $U = (0,L)^n = n - \dim$ coordinate cell. In this case we impose periodic boundary conditions

$$\begin{aligned} \vec{u}(\vec{x} + L\vec{e}_i) &= \vec{u}(\vec{x}) \qquad \forall\vec{x} \quad 1 \le i \le n, \\ p(\vec{x} + L\vec{e}_i) &= p(\vec{x}) \qquad \forall\vec{x}. \end{aligned}$$

Setting (c) is simpler than (b) but maintains most of the relevant features of a problem on a bounded set. For considering the problem in this setting we let

$$H^n = \left\{\vec{u} \in L^2(U)^n : \vec{u} \text{ satisfies the periodic boundary conditions}\right\}$$

$$(\vec{u},\vec{v})_{H^n} = \int_U \vec{u}(\vec{x}) \cdot \vec{v}(\vec{x})\, dx$$

and we let H denote H^n in the case n=1. For convenience, we let $L = 2\pi,\ n = 2$. Then an orthonormal basis for H is the family

$$\phi_{mn}(x,y) = e^{imx} e^{iny}$$

where the normalization constants have been suppressed for convenience. Then

$$\vec{f}(x,y) = \sum_{m,n}(f_{mn}\vec{e}_1 + g_{mn}\vec{e}_2)\ \phi_{mn}(x,y)$$

$$p(x,y) = \sum_{m,n} p_{mn}\ \phi_{mn}(x,y)$$

$$\vec{u}(x,y) = \sum_{m,n}(u_{mn}\vec{e}_1 + v_{mn}\vec{e}_2)\ \phi_{mn}(x,y)$$

where

$$p_{mn} = (p,\phi_{mn})_H$$

$$and \qquad (u_{mn}\vec{e}_1 + v_{mn}\vec{e}_2) = (\vec{u},\phi_{mn})_{H}., \qquad (f_{mn}\vec{e}_1 + g_{mn}\vec{e}_2) = (\vec{f},\phi_{mn})_H.$$

Then

$$\nabla p = \sum_{m,n} p_{mn}\ \nabla\phi_{mn}(x,y) = \sum_{m,n} p_{mn}\ \ i\,(m\vec{e}_1 + n\vec{e}_2)\phi_{mn}(x,y)$$

$$div\ \vec{u} = \vec{u}(x,y) = \sum_{m,n}(u_{mn}\partial_x\phi_{mn}(x,y) + v_{mn}\partial_y\phi_{mn}(x,y))$$

$$= \sum_{m,n} i\,(m\,u_{mn} + n\,v_{mn})\ \phi_{mn}(x,y),$$

and

$$\nabla^2\vec{u} = \sum_{m,n}(u_{mn}\vec{e}_1 + v_{mn}\vec{e}_2)\ \nabla^2\phi_{mn}(x,y)$$

$$= -\sum_{m,n}(m^2+n^2)(u_{mn}\vec{e}_1 + v_{mn}\vec{e}_2)\ \phi_{mn}(x,y).$$

The Stokes system then becomes (letting $\nu = 1$ for convenience),

$$-\nabla^2\vec{u} + \nabla p = \sum_{m,n}[(m^2+n^2)(u_{mn}\vec{e}_1 + v_{mn}\vec{e}_2) + p_{mn}\ \ i\,(m\vec{e}_1 + n\vec{e}_2)]\ \phi_{mn}(x,y)$$

$$= \sum_{m,n}(f_{mn}\vec{e}_1 + g_{mn}\vec{e}_2)\ \phi_{mn}(x,y),$$

and

$$\sum_{m,n} i\,(m\,u_{mn} + n\,v_{mn})\ \phi_{mn}(x,y) = 0.$$

This reduces then to the following algebraic system

$$(m^2+n^2)\,u_{mn} + i\,m\,p_{mn} = f_{mn}$$

$$(m^2+n^2)\,v_{mn} + i\,n\,p_{mn} = g_{mn}$$

$$m\,u_{mn} + n\,v_{mn} = 0.$$

It follows that

$$i(m^2+n^2)p_{mn} = m\,f_{mn} + n\,g_{mn}$$

or

$$p_{mn} = \frac{m\,f_{mn} + n\,g_{mn}}{i(m^2+n^2)}$$

and then

$$u_{mn} = \frac{f_{mn}}{m^2+n^2} - \frac{m}{m^2+n^2}\left(\frac{m\,f_{mn} + n\,g_{mn}}{i(m^2+n^2)}\right).$$

It is clear that $u_{00} = \left(\vec{u},\vec{1}\right)_H,\ \ and\ \ p_{00} = (p,1)_H$ are indeterminate from these equations, which is to say, $\vec{u}\ and\ p$ are not uniquely determined but are unique only up to additive constants. Then, in the quotient spaces

$$\tilde{H} = \{p \in H : (p,1)_H = 0\}\ \ and\ \ \tilde{H}^n = \left\{\vec{u} \in H^n : \left(\vec{u},\vec{1}\right)_H = 0\right\}$$

the solutions of the Stokes system (1) are uniquely determined.

In addition, using the isomorphism

$$H^2 \ni \vec{f} \ \leftrightarrow\ \{f_{mn}, g_{mn}\} \in (\ell_2)^2$$

we can show that for every $f \in H^2$, there is a "unique" pair $(\vec{u},p)$ such that $\nabla^2\vec{u} \in H^2\ and\ \nabla p \in H^2$. This means that there exists a one to one mapping from $\tilde{H}^2 \times \tilde{H}$ onto H^2.

On the basis of this example, we now feel justified is seeking an abstract setting in which to prove the existence of a weak solution to the Stokes system in the more general situation of case (b). If we are successful in this attempt it will hopefully lead to the correct approach for dealing with the more difficult issues of the nonlinear steady and unsteady equations.

Weak Formulation of the Stokes System

Let $U \subset R^n$ $(n = 2,3)$ be open and bounded with a smooth boundary ∂U. Let $\vec{f} \in L^2(U)^n$ be given and consider

$$\begin{aligned} -\nu\nabla^2\vec{u} + \nabla p &= \vec{f} \qquad in\ U \subset R^n \\ div\ \vec{u} &= 0, \qquad in\ U, \qquad\qquad (S) \\ \vec{u} &= 0 \qquad on\ \partial U. \end{aligned}$$

This set of $(n+1)$ equations for the $(n+1)$ unknowns $\vec{u},p$ is called the Stokes system. We will now define several function spaces and develop their properties in preparation for giving a weak formulation for the Stokes system.

Function Spaces

Let

$$E(U) = \left\{\vec{u} \in L^2(U)^n : div\ \vec{u} \in L^2(U)\right\}$$

with

$$(\vec{u},\vec{v})_E = \int_U \vec{u}(x) \cdot \vec{v}(x)\,dx + \int_U div\,\vec{u}(x)\ div\,\vec{v}(x)\,dx$$

and

$$\|\vec{u}\|_E^2 = \|\vec{u}\|_{L^2(U)^n}^2 + \|div\ \vec{u}\|_{L^2(U)}^2.$$

Then we have the following results about $E(U)$

Lemma 1 (a) $E(U)$ is a Hilbert space
(b) $C^\infty(U)^n \subset E(U)$ is dense

Evidently $E(U)$ is a vector valued version of $H^1(U)$ and the proof of lemma 1 is nearly identical to the proof of analogous results about $H^1(U)$.

Recall that the "trace operator" $T_0 : H^1(U) \to L^2(\partial U)$ had the interpretation of a restriction to the boundary operator. In the simplest case, $U = R^n_+,\ \partial U = R^{n-1},$ we have for any $\phi \in C^1(\bar{U})\ and\ \vec{x} = (\vec{x}', x_n),\ \vec{x}' \in R^{n-1},\ x_n > 0,$

$$|\phi(\vec{x}',0)|^2 = -\int_0^\infty \partial_n\left(|\phi(\vec{x}',x_n)|^2\right)dx_n.$$

Then

$$\int_{R^{n-1}} |\phi(\vec{x}',0)|^2 dx' = -\int_{R^{n-1}}\int_0^\infty \partial_n\left(|\phi(\vec{x}',x_n)|^2\right)dx_n\,dx'$$

and

$$\|\phi(\vec{x}',0)\|^2_{L^2(R^{n-1})} \leq 2\|\partial_n\phi\|_{L^2(R^n_+)}\|\phi\|_{L^2(R^n_+)} \leq \|\partial_n\phi\|^2_{L^2(R^n_+)} + \|\phi\|^2_{L^2(R^n_+)}$$

i.e., $\|T_0[\phi]\|^2_{L^2(R^{n-1})} \leq \|\phi\|^2_{H^1(R^n_+)}$.

This proves that the restriction operator is continuous from $H^1(U)$ *to* $L^2(\partial U)$ in this simple case. Extension to more general sets U is accomplished using the previously discussed local charts for "flattening the boundary"..

Recall further that T_0 satisfied

i) $\forall\phi \in C^1(\bar{U}) \qquad T_0\phi = \phi|_{\partial U}$
ii) $\ker T_0 = H^1_0(U)$
iii) $Rng\, T_0 = H^{1/2}(\partial U)$

Here $\qquad H^{1/2}(\partial U) \subset L^2(\partial U) \subset H^{-1/2}(\partial U)V$

where $\qquad \langle f,g\rangle_{H^{-1/2}\times H^{1/2}} = (f,g)_{L^2(\partial U)} = \int_{\partial U} f\,g\;ds;$

i.e., the duality pairing on $H^{-1/2}(\partial U)\times H^{1/2}(\partial U)$ is just the extension of the $L^2(\partial U)$ inner product. This makes $H^{-1/2}(\partial U)$ a realization for the dual of $H^{1/2}(\partial U)$. Our aim now is to define a similar restriction operator for the space $E(U)$.

Trace Theorem for $E(U)$ Suppose $U \subset R^n$ has smooth boundary ∂U with $\vec{n}_x$ =outward unit normal to ∂U *at* x. Then there is a continuous linear map

$$T:\ E(U) \to H^{-1/2}(\partial U)\quad (onto)$$

with $\qquad T\vec{w} = \vec{w}\cdot\vec{n}_x|_{x\in\partial U} \qquad \forall\vec{w} \in C^1(\bar{U})^n$

Moreover, $\quad \forall\vec{u} \in E(U),\ w \in H^1(U)$

$$(\vec{u},\nabla w)_{L^2(U)^n} + \left(div\ \vec{u},w\right)_{L^2(U)} = \langle T\vec{u},T_0w\rangle_{H^{-1/2}\times H^{1/2}}$$

To see the meaning of this last equation suppose we have $\left\{\vec{\phi}_n,\psi_n\right\} \in C^1(\bar{U})^n \times C^1(\bar{U})$, and note that a version of the divergence theorem asserts that for each n, we have the following integration by parts formula,

$$\int_U \vec{\phi}_n\cdot\nabla\psi_n\,dx = \int_{\partial U}\vec{\phi}_n\cdot\vec{n}_x\ \psi_n ds - \int_U div\ \vec{\phi}_n\,\psi_n dx \qquad \forall n$$

Suppose also that as n tends to infinity,

$$\begin{array}{ccccc}\int_U \vec{\phi}_n\cdot\nabla\psi_n\,dx & + & \int_U div\ \vec{\phi}_n\,\psi_n dx & = & \int_{\partial U}\vec{\phi}_n\cdot\vec{n}_x\ \psi_n ds\\ \downarrow & & \downarrow & & \downarrow\\ \int_U \vec{u}\cdot\nabla w\,dx & + & \int_U div\ \vec{u}\,w\ dx & = & \int_{\partial U}\vec{u}\cdot\vec{n}_x\ w\ ds\end{array}$$

Note that by the continuity of the inner products

$$\left\{\begin{array}{c}\vec{\phi}_n \to \vec{u}\quad in\ \ L_2(U)^n\\ \nabla\psi_n \to \nabla w\quad in\ \ L_2(U)\end{array}\right\} \qquad \text{implies} \qquad \int_U \vec{\phi}_n\cdot\nabla\psi_n\,dx \to \int_U \vec{u}\cdot\nabla w\,dx$$

Similarly,

$$\left\{ \begin{array}{c} div\,\vec{\phi}_n \to div\,\vec{u} \quad in \quad L_2(U) \\ \psi_n \to w \quad in \;\; L_2(U) \end{array} \right\} \qquad \text{implies} \qquad \int_U div\vec{\phi}_n\,\psi_n\,dx \to \int_U div\,\vec{u}\,w\,dx$$

In addition,

$$\vec{\phi}_n \to \vec{u} \quad in \;\; L_2(U)^n \quad and \quad div\,\vec{\phi}_n \to div\,\vec{u} \quad in \quad L_2(U)$$
$$implies \qquad \left\| \vec{\phi}_n - \vec{u} \right\|_{E(U)} \to 0, \;\; and \quad \vec{u} \in E(U)$$

and

$$\psi_n \to w \quad in \;\; L_2(U) \quad and \quad \nabla\psi_n \to \nabla w \quad in \;\; L_2(U)$$
$$implies \quad \|\psi - w\|_1 \to 0 \quad and \quad w \in H^1(U)$$

Together, these results imply that $\vec{\phi}_n \to \vec{u}, (in\ E(U)), \quad and\ \psi_n \to w\ \left(in\ H^1(U)\right)$ and

$$\int_{\partial U} \vec{\phi}_n \bullet \vec{n}_x \;\; \psi_n ds \to \langle T\vec{u}, T_0 w\rangle_{H^{-1/2}\times H^{1/2}} = \int_{\partial U} \vec{u} \bullet \vec{n}_x \;\; w\,ds$$

where the trace operator T is defined in terms of this limit.

To show that T maps $E(U)$ $onto$ $H^{-1/2}(\partial U)$, pick an arbitary $\psi \in H^{-1/2}(\partial U)$, and let

$$\phi = \psi - \frac{\langle \psi, 1\rangle}{\langle 1, 1\rangle}$$

Then $\phi \in H^{-1/2}(\partial U)$ and $\langle \phi, 1\rangle = 0$; i.e.,we subtract from ψ its average over ∂U to obtain a ϕ whose integral over ∂U is zero. This condition on ϕ is sufficient to imply the existence of a (nonunique) $p \in H^1(U)$ satisfying

$$\nabla^2 p = 0 \qquad in \quad U$$
$$\partial_N p = \nabla p \bullet \vec{n}_x = \phi.$$

Denote the solution of this Neumann problem by $p = p(\phi)$, and let

$$\vec{u} = \nabla p(\phi) + \frac{\langle \psi, 1\rangle}{\langle 1, 1\rangle}\vec{u}_0$$

where

$\vec{u}_0 \in C^1(\bar{U})^n$ is such that $T\vec{u}_0 = 1$

Then $\qquad \vec{u} \in L^2(U)^n \quad and \quad div\;\vec{u} = \nabla^2 p + \dfrac{\langle \psi, 1\rangle}{\langle 1, 1\rangle} div\vec{u}_0 \in L^2(U)$

which means $\vec{u} \in E(U)$. Moreover,

$$T\vec{u} = \partial_N p(\phi) + \frac{\langle \psi, 1\rangle}{\langle 1, 1\rangle} = \phi + \frac{\langle \psi, 1\rangle}{\langle 1, 1\rangle} = \psi.$$

Thus for every $\psi \in H^{-1/2}(\partial U)$ there is a $\vec{u} \in E(U)$ such that $T\vec{u} = \psi$.

Now let $E_0(U) =$ the completion of $C_c^\infty(U)^n$ in the norm of $E(U)$. Then $E_0(U) = \ker T$. To see that this is true, let $\vec{u} \in E_0(U)$ $with$ $\left\{\vec{\phi}_n\right\} \in C_c^\infty(U)^n$ such that $\|\vec{\phi}_n - \vec{u}\|_E \to 0$ as $n \to \infty$. Then $T\vec{\phi}_n = 0\ \forall n$ which implies $T\vec{u} = 0$ since T is continuous.

Conversely, suppose $\vec{u} \in E(U)$ is such that $T\vec{u} = 0$. For $F \in C_c^\infty(R^n)$, let $f =$ the restriction of F to U. Since $T\vec{u} = 0$, we have

$$\langle T\vec{u}, T_0 F\rangle = 0.$$

But

$$\langle T\vec{u}, T_0 F\rangle = \int_U (f\, div\vec{u} + \vec{u} \cdot \nabla f)\, dx.$$

Then $\int_{R^n} (F\ [div\ \vec{u}]^{\sim} + [\vec{u}]^{\sim} \cdot \nabla F)\, dx = 0 \qquad \forall F \in C_c^\infty(R^n),$

where $[div\ \vec{u}]^{\sim}$ *and* $[\vec{u}]^{\sim}$ denote the extensions by zero from U to all of R^n. But

$$\int_{R^n} F\ div[\vec{u}]^{\sim} dx = -\int_{R^n} [\vec{u}]^{\sim} \cdot \nabla F\, dx \qquad \forall F \in C_c^\infty(R^n),$$

which, combined with the previous result, implies that $[div\ \vec{u}]^{\sim} = div[\vec{u}]^{\sim}$. Then it follows that $[\vec{u}]^{\sim} \in E(R^n)$. It now has to be shown that this forces $\vec{u} \in E_0(U)$ but the key part of the result resides in what we have just shown. More precisely, when $T\vec{u} = 0$ we can extend $\vec{u}$ to the whole space by letting $\vec{u} = 0$ outside U and the extended function is still an element of $H^1(R^n)^n$. In general, when extending a function in $H^1(U)^n$ by zero, the extended function is no longer in $H^1(R^n)^n$.

Summarizing, we have a continuous linear map

$$T:\ \ E(U) \to H^{-1/2}(\partial U) \quad (onto)$$

with $\qquad T\vec{w} = \vec{w} \cdot \vec{n}_x|_{x\in\partial U} \qquad \forall \vec{w} \in C^1(\bar{U})^n$

$$\ker T = E_0(U)$$

and $\qquad (\vec{u}, \nabla w)_{L^2(U)^n} + \left(div\ \vec{u}, w\right)_{L^2(U)} = \langle T\vec{u}, T_0 w\rangle_{H^{-1/2}\times H^{1/2}} \qquad \forall \vec{u} \in E(U),\ w \in H^1(U)$

In order to attack the Stokes system, our plan is going to be to incorporate the continuity equation $div\vec{u} = 0$ into the definition of the solution space. To this end, we begin by considering

$$K = \left\{\vec{\phi} \in C_c^\infty(U)^n :\ div\vec{\phi} = 0\right\}.$$

This is a linear space but it cannot support Hilbert space, Banach space or even metric space topology without destroying the character of the space. Note, however, that for any $\vec{\phi} \in K$, and any distribution $p \in D'(U)$, we have

$$\left\langle \nabla p, \vec{\phi}\right\rangle_{D_n' \times D_n} = \sum_{i=1}^n \langle \partial_i p, \phi_i\rangle_{D'\times D} = -\sum_{i=1}^n \langle p, \partial_i \phi_i\rangle_{D'\times D} = -\left\langle p, div\vec{\phi}\right\rangle_{D'\times D} = 0;$$

i.e., this shows that $\qquad div : D(U)^n \ni \vec{u} \ \to\ div\,\vec{u} \in D(U)$

and $\qquad grad : D'(U)^n \ni \nabla p \ \leftarrow\ p \in D'(U)$

are transposes of one another. A consequence of this fact is that every gradient annihilates K but the converse, that every annihilator of K must be a gradient, is much more difficult to prove. The following result will be stated without proof.

Theorem 1 Suppose $U \subset R^n$ is open, bounded and has Lipschitz smooth boundary ∂U.

a. For $\vec{f} \in D'(U)^n \quad \langle \vec{f}, \vec{\phi} \rangle = 0 \quad \forall \vec{\phi} \in K \Leftrightarrow \vec{f} = \nabla p \text{ some } p \in D'(U)$

b. For $\vec{f} \in H^{-1}(U)^n \quad \langle \vec{f}, \vec{\phi} \rangle = 0 \quad \forall \vec{\phi} \in K \Leftrightarrow \vec{f} = \nabla p \text{ some } p \in L^2(U)$

c. For $\vec{f} \in L^2_{loc}(U)^n \quad \langle \vec{f}, \vec{\phi} \rangle = 0 \quad \forall \vec{\phi} \in K \Leftrightarrow \vec{f} = \nabla p \text{ some } p \in H^1(U)$

Note that $\quad grad : L^2(U) \to H^{-1}(U)^n$

is not an isomorphism since the gradient is not injective on any space containing the constants. However, the gradient is an isomorphism on the quotient space

$$L^2_*(U) = L^2(U)\,\mathrm{mod}(R) \cong \left\{ p \in L^2(U) : \int_U p\,dx = 0 \right\}$$

Now define

$$H = \text{completion of } K \text{ in the norm of } L^2(U)^n$$

$$V = \text{completion of } K \text{ in the norm of } H^1(U)^n$$

where

$$\left|\left|\vec{\phi}\right|\right|^2_{H^1(U)^n} = \sum_{i=1}^n \|\phi_i\|^2_{L^2(U)} + \sum_{i=1}^n \sum_{j=1}^n \|\partial_j \phi_i\|^2_{L^2(U)}$$

Since K consists of the divergence free vector valued test functions, we expect the elements of H and V to continue to have some of the properties enjoyed by elements of K.

Theorem 2 Suppose $U \subset R^n$ is open, bounded and has Lipschitz smooth boundary $\partial U,$ and let

$$H_* = \left\{ \vec{u} \in L^2(U)^n : div\,\vec{u} = 0, \text{ and } T\vec{u} = 0 \right\}$$

$$\hat{H} = \left\{ \vec{u} \in L^2(U)^n : \vec{u} = \nabla p, \text{ for some } p \in H^1(U) \right\}.$$

Then $\quad H = H_* \text{ and } \hat{H} = H^\perp.$

Proof- (a) $\hat{H} \subset H^\perp$
Suppose $\vec{u} \in \hat{H}.$ Then $\quad \left(\left(\vec{u}, \vec{\phi}\right)\right)_0 = \left(\left(\nabla p, \vec{\phi}\right)\right)_0 = -\left(p, div\,\vec{\phi}\right)_0 = 0 \quad \forall \vec{\phi} \in K.$

Here we use $((\cdot,\cdot))_0$ *and* $(\cdot,\cdot)_0$ to denote the inner products in $L^2(U)^n$ *and* $L^2(U),$ respectively. Since K is dense in H, it follows from this result and the continuity of the inner product, that

$$((\vec{u}, \vec{v}))_0 = 0 \quad \forall \vec{v} \in H,$$

hence $\hat{H} \subset H^\perp.$

(b) $H^\perp \subset \hat{H}.$
Suppose $\vec{u} \in H^\perp.$ Then $\quad \left(\left(\vec{u}, \vec{\phi}\right)\right)_0 = 0 \quad \forall \vec{\phi} \in K \subset H,$

and it follows from theorem 1 that $\vec{u} = \nabla p$ for some $p \in D'(U).$ But $\nabla p = \vec{u} \in L^2(U)^n$ so $p \in H^1(U),$ and it follows that $\vec{u} \in \hat{H}.$

The results (a) and (b) together imply $\hat{H} = H^{\perp}$.

(c) $H \subset H_*$.

Suppose $\vec{u} \in H$. Then $\vec{u}$ is the limit in $L^2(U)^n$ of a sequence $\{\vec{\phi}_n\} \subset K$. Then $div\vec{\phi}_n = 0 \quad \forall n$ implies $div\vec{u} = 0 \ in \ L^2(U)$. Moreover, since $div\vec{u} = div\vec{\phi}_n = 0 \in L^2(U)$, we have $\vec{u},\vec{\phi} \in E(U)$ and hence

$$\left|\left|\vec{u} - \vec{\phi}_n\right|\right|_{E(U)} = \left|\left|\vec{u} - \vec{\phi}_n\right|\right|_{L^2(U)^n} \to 0 \ as \ n \to \infty$$

i.e.,

$$\vec{\phi}_n \to \vec{u} \ in \ E(U).$$

But this implies that $0 = T\vec{\phi}_n \to T\vec{u} = 0$, since T is continuous on $E(U)$. This proves $\vec{u} \in H \ implies \ \vec{u} \in H_*; \ i.e., \ H \subset H_*$.

To see that $H = H_*$, suppose H is not all of H_*. then H, since it is the $L^2(U)^n$ closure of K, is a closed subspace of H_*. Then H has an orthogonal complement, H^0, in H_*. But if $\vec{u} \in H^0$, then $\vec{u} = \nabla p$ for some $p \in H^1(U)$ by the argument used to prove (b). In this case, $\vec{u} \in H^0 \subset H_*$ satisfies

$$\nabla^2 p = div\vec{u} = 0 \quad and \quad T\vec{u} = \partial_N p = 0$$

Then p =constant and $\vec{u} = \nabla p = 0$, so $H^0 = \{0\} \ and \ H = H_*$.■

A more precise characterization of $H^{\perp}$ is possible when ∂U has additional smoothness.

Theorem 3 Suppose $U \subset R^n$ is open, bounded and has a smooth boundary ∂U. Then

$$L^2(U)^n = H \oplus H^{\perp} = H \oplus H_1 \oplus H_2$$

where

$$H_1 = \left\{\vec{u} \in L^2(U)^n : \vec{u} = \nabla p, \ p \in H^1(U), \ \nabla^2 p = 0\right\}$$
$$H_2 = \left\{\vec{u} \in L^2(U)^n : \vec{u} = \nabla p, \ p \in H^1_0(U)\right\}$$

Proof- It is clear that $H_1, H_2 \subset H^{\perp}$. To see that $H_1 \perp H_2$, let $\vec{u} = \nabla p \in H_1 \ and \ \vec{v} = \nabla q \in H_2$. Then

$$((\vec{u},\vec{v}))_0 = ((\vec{u},\nabla q))_0 = \langle T\vec{u}, T_0 q\rangle - (div\vec{u}, q)_0.$$

But

$$\langle T\vec{u}, T_0 q\rangle = 0 \ \ for \ q \in H^1_0(U) \ \ and \ \ (div\vec{u}, q)_0 = 0 \ as \ div\vec{u} = \nabla^2 p = 0.$$

Now, to show $L^2(U)^n = H \oplus H_1 \oplus H_2, \ let \ \vec{u} \in L^2(U)^n$. Then $div\vec{u} \in H^{-1}(U)$ so our results on the existence of weak solutions to elliptic boundary value problems imply that there exists a unique $p \in H^1_0(U) \ such \ that \ \nabla^2 p = div\vec{u} \in H^{-1}(U)$.

Let $\vec{u}_2 = \nabla p$, and note that $\vec{u}_2 \in H_2$. Then

$$div\vec{u} - \nabla^2 p = div(\vec{u} - \nabla p) = 0,$$

so $\quad \vec{u} - \nabla p \in E(U),$

$$and \quad T(\vec{u} - \nabla p) \in H^{-1/2}(\partial U).$$

Next, let q denote the weak solution of

$$\nabla^2 q = 0 \quad in \quad U,$$
$$\partial_N q \;= T(\vec{u} - \nabla p) \in H^{-1/2}(\partial U).$$

Since the Green's formula implies,

$$\langle T(\vec{u} - \nabla p), 1\rangle = \int_U 1 div(\vec{u} - \nabla p) = 0,$$

it follows that the Neumann problem has a (non-unique) solution $q \in H^1(U)$. Finally, we let $\vec{u}_1 = \nabla q$ so that $\vec{u}_1 \in H_1$ and $\vec{u}_0 = \vec{u} - \vec{u}_1 - \vec{u}_2 \in H$; $i.e.$,

$$div\, \vec{u}_0 = div(\vec{u} - \vec{u}_1 - \vec{u}_2) = div\vec{u} - \nabla^2 q - \nabla^2 p = div(\vec{u} - \nabla p) = 0$$

$$T\vec{u}_0 = T(\vec{u} - \vec{u}_1 - \vec{u}_2) = T(\vec{u} - \nabla p) - \partial_n q = 0.$$

This shows that every $\vec{u} \in L^2(U)^n$ can be written $\vec{u} = \vec{u}_0 + \vec{u}_1 + \vec{u}_2$.■

The Hilbert space projection theorem implies the existence of projections

$$P_H : L^2(U)^n \to H \qquad and \qquad Q_H : L^2(U)^n \to H^{\perp}$$

where

$$P_H\vec{u} = \vec{u}_0 = \vec{u} - \vec{u}_1 - \vec{u}_2.$$

In addition, P_H maps $H_0^1(U)^n$ continuously into $H^1(U)^n$. To see this, suppose $\vec{u} \in H_0^1(U)^n$. Then

$$\nabla^2 q = div\, \vec{u} \in L^2(U)$$
$$q \in H_0^1(U),$$

has a unique solution $q \in H^2(U),\ and\ \vec{u}_2 = \nabla q \in H^1(U)^n$.
In addition,

$$\vec{u} - \vec{u}_2 \in H^1(U)^n,\ \ so\ \ T(\vec{u} - \vec{u}_2) \in H^{1/2}(\partial U)$$

so

$$\nabla^2 p = 0 \qquad in\ U$$

$$\partial_N p = T(\vec{u} - \vec{u}_2),$$

has a solution $p \in H^2(U),\ and\ \vec{u}_1 = \nabla p \in H^1(U)^n$. Then

$$\vec{u}_0 = \vec{u} - \vec{u}_1 - \vec{u}_2 \in H^1(U)^n$$

and

$$P_H : H_0^1(U)^n \to H^1(U)^n$$

is bounded since

$$\|P_H\vec{u}\|_{H^1(U)^n} = \|\vec{u}_0\|_{H^1(U)^n} \leq \|\vec{u}\|_{H^1(U)^n}$$

Next, we will give a characterization for the space V obtained by completing K in the $H^1(U) - norm.$

Theorem 4 Suppose $U \subset R^n$ is open, bounded and has Lipschitz smooth boundary ∂U, and let

$$V_* = \{\vec{u} \in H_0^1(U)^n \ : \ div\vec{u} = 0\}.$$

Then $\quad V_* = V = H^1(U)^n \ completion \ of \ K = \left\{\vec{\phi} \in C_c^\infty(U)^n \ : \ \ div\vec{\phi} = 0\right\}$

Proof- (a) $V \subset V_*$
Suppose $\vec{u} \in V.$ Then $\vec{u}$ is the limit in $H^1(U)^n$ of a sequence $\{\vec{\phi}_n\} \subset K.$ This implies

$$\left|\left|\vec{\phi}_n - \vec{u}\right|\right|_{H^1(U)^n} \to 0 \qquad and \qquad 0 = div\vec{\phi}_n \to div\vec{u} \ \ in \ L_2(U),$$

hence

$$\vec{u} \in H_0^1(U)^n \qquad\qquad and \qquad\qquad div\,\vec{u} = 0 \ in \ \ L^2(U).$$

i.e., $\qquad \vec{u} \in V \ \ implies \ \ \vec{u} \in V_* \ \ so \ \ V \subset V_*.$

To now show that V is, in fact, all of $V_*,$ we will show that any continuous linear functional L on V_* which vanishes on V must then vanish on all of $V_*.$ So, suppose L is such a linear functional on V_*. Since V_* is, by its definition, a closed subspace of $H_0^1(U)^n$, it follows that any linear functional that is continuous on V_* can be extended to $H_0^1(U)^n$ as a continuous linear functional. Of course this extension is not unique but, since the dual of $H_0^1(U)^n$ is $H^{-1}(U)^n,$ the extended functional has a representation of the form

$$L[\bullet] = \sum_{i=1}^{n}\langle\lambda_i, \bullet\rangle \qquad \lambda_i \in H^{-1}(U).$$

Since we have assumed that L vanishes on $V = H^1(U)^n \ completion \ of \ K,$ it follows that

$$L\left[\vec{\phi}\right] = \sum_{i=1}^{n}\langle\lambda_i, \phi_i\rangle = 0 \qquad \forall\vec{\phi} \in K.$$

But then theorem 1 implies that $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) = \nabla p \ for \ some \ p \in L^2(U).$ Then this implies, in turn, that

$$L[\vec{v}] = \sum_{i=1}^{n}\langle\lambda_i, v_i\rangle = -\langle p, div\,\vec{v}\rangle = -(p, div\,\vec{v})_0 = 0 \qquad \forall\vec{v} \in V_*.;$$

i.e., we have succeeded in showing that an L that vanishes on V necessarily also vanishes on $V_*.$■

If U is bounded but the boundary is not Lipschitz smooth, then it is not known whether $V = V_*.$ If U is not bounded, then there are examples where $\dim(V\backslash V_*) = 1$ or even larger. When V is not equal to $V_*.$ this can lead to serious difficulties in the weak formulation of the Stokes problem.

Weak Solutions of the Stokes System

We now return to considering the weak formulation of the Stokes system (S). If $\{\vec{u}, p\}$ is a solution for (S), then for any $\vec{\phi} \in K,$

$$-\nu\left(\left(\nabla^2\vec{u}, \vec{\phi}\right)\right)_0 = -\nu\int_U \nabla^2\vec{u} \bullet \vec{\phi}\,dx$$

$$= \nu \int_U \nabla\vec{u} * \nabla\vec{\phi}\, dx =: \nu\left[\left[\vec{u},\vec{\phi}\right]\right];$$

i.e.,

$$-\int_U \nabla^2\vec{u} \bullet \vec{\phi}\, dx = -\int_U \sum_{j=1}^{n} \phi_j \nabla^2 u_j\, dx$$

$$= \int_U \sum_{j=1}^{n} \nabla\phi_j \bullet \nabla u_j\, dx = \int_U \nabla\vec{\phi} * \nabla\vec{u}\, dx = \left[\left[\vec{u},\vec{\phi}\right]\right].$$

Also

$$\left(\left(\nabla p,\vec{\phi}\right)\right)_0 = \int_U \nabla p \bullet \vec{\phi}\, dx = -\int_U p\, div\vec{\phi}\, dx + \int_{\partial U} T_0 p\, T\vec{\phi}\, dS = 0.$$

Thus $\qquad \nu[\left[\vec{u},\vec{\phi}\right]] = ((\vec{f},\vec{\phi}))_0 \qquad \forall\vec{\phi} \in K,$

and $\vec{u} \in V = \{\vec{u} \in H_0^1(U)^n : div\vec{u} = 0\}$ is a weak solution of the Stokes system if

$$\nu\left[\left[\vec{u},\vec{\phi}\right]\right] = ((\vec{f},\vec{\phi}))_0 \qquad \forall\vec{\phi} \in V.$$

Conversely, suppose $\vec{u} \in V$ is a weak solution of the Stokes system. Then

$$\vec{u} \in V \quad implies \quad u_i \in H_0^1(U),\ 1 \le i \le n,\ hence \quad T_0 u_i = u|_{\partial U} = 0.$$
$$\text{and} \qquad div\ \vec{u} = 0.$$

Moreover,

$$\nu\left[\left[\vec{u},\vec{\phi}\right]\right] = ((\vec{f},\vec{\phi}))_0 \qquad \forall\vec{\phi} \in V.$$

implies

$$((-\nu\nabla^2\vec{u} - \vec{f},\vec{\phi}))_0 = 0 \qquad \forall\vec{\phi} \in K.$$

Then $\qquad -\nu\nabla^2\vec{u} - \vec{f} \in K^0$ and it follows from theorem 1 that, $-\nu\nabla^2\vec{u} - \vec{f} = \nabla p$

$$for\ some\ p \in H^1(U)\ if \quad -\nu\nabla^2\vec{u} - \vec{f} \in L^2_{loc}(U)^n$$
or $for\ some\ p \in L^2(U)\ if \quad -\nu\nabla^2\vec{u} - \vec{f} \in H^{-1}(U)^n.$

Here p is uniquely determined in $L^2_*(U)$ but otherwise there is an arbitrary additive constant. We seem to have proved,

Theorem 5 (Weak Formulation of Stokes System)

Suppose $U \subset R^n$ is open, bounded and has Lipschitz smooth boundary ∂U. Then the following are equivalent:

1. $\vec{u} \in V \ \ satisfies \quad \nu\left[\left[\vec{u},\vec{\phi}\right]\right] = ((\vec{f},\vec{\phi}))_0 \qquad \forall\vec{\phi} \in V.$

2. i) $\vec{u} \in H_0^1(U)^n$ so $\vec{u}$ vanishes on ∂U in the trace sense

ii) $\exists\ p \in L^2_*(U)\ \ such\ that\ -\nu\nabla^2\vec{u} - \vec{f} = \nabla p \ \ in\ D'(U)^n$

iii) $div\ \vec{u} = 0$

For $U \subset R^n$ open, bounded with a Lipschitz smooth boundary $\partial U,$ the Poincare inequality holds for $H_0^1(U)^n$ and it follows that

$$\int_U \sum_{j=1}^{n} \nabla v_j \nabla u_j \, dx = \int_U \nabla\vec{v} \cdot \nabla\vec{u} \, dx = \left[\left[\vec{u},\vec{\phi}\right]\right]$$

defines an inner product on V, and that V is a Hilbert space for the associated norm. Then we have

Theorem 6 (Existence of a Weak Solution of Stokes System)
Suppose $U \subset R^n$ is open, bounded and has Lipschitz smooth boundary ∂U. Then for each $\vec{f} \in L^2(U)^n$ there exists a unique $\vec{u} \in V$ such that $\vec{u}$ is a weak solution of the Stokes system. In addition there exists a $p \in L^2(U)$ satisfying $\nabla p = -\nu\nabla^2\vec{u} - \vec{f}$ and p is unique up to an additive constant.

Proof- For $\vec{f} \in L^2(U)^n$, $let\ F(\vec{v}) = ((\vec{f},\vec{v}))_0 \ \forall\vec{v} \in V.$ Then

$$|(F(\vec{v}))| \le \|\vec{f}\|_0\|\vec{v}\|_0 \le \|\vec{f}\|_0\|\vec{v}\|_V \quad \forall\vec{v} \in V.$$

Then F is a bounded linear functional on V and it follows that there exists a unique $\vec{z}_F \in V$ such that

$$F(\vec{v}) = [[\vec{z}_F,\vec{v}]] \qquad \forall\vec{v} \in V.$$

Then $\qquad \nu[[\vec{u},\vec{v}]] = F(\vec{v}) = [[\vec{z}_F,\vec{v}]] \qquad \forall\vec{v} \in V,$
implies

$$\vec{u} = \frac{1}{\nu}\vec{z}_F \in V$$

is the unique weak solution of the Stokes system. It is clear that $div\ \vec{u} = 0$ by virtue of the fact that $\vec{u} \in V$ and $\nu[[\vec{u},\vec{v}]] = F(\vec{v}) \quad \forall\vec{v} \in V,$ implies

$$\left[\left[-\nu\nabla^2\vec{u} - \vec{f},\vec{\phi}\right]\right] = 0 \qquad \forall\vec{\phi} \in K,$$

which in turn implies that $-\nu\nabla^2\vec{u} - \vec{f} = \nabla p\ for\ \ p \in D'(U).$ Additionally, $\vec{u} \in V$ implies $-\nu\nabla^2\vec{u} - \vec{f} \in H^{-1}(U)^n$ which is sufficient to conclude that $p \in L^2(U)$ ■

Corollary to Theorem 5 Assertions 1 and 2 are equivalent to

3. $\vec{u} \in V$ minimizes $\qquad J(\vec{u}) = \frac{1}{2}\nu[[\vec{u},\vec{u}]] - ((\vec{f},\vec{u}))_0$

Proof- For $\vec{u},\vec{v} \in V$

$$J(\vec{u} + t\vec{v}) = J(\vec{u}) + t\left(\nu[[\vec{u},\vec{v}]] - ((\vec{f},\vec{v}))_0\right) + \frac{1}{2}t^2\nu\,[[\vec{v},\vec{v}]].$$

Then 1 implies $J(\vec{u} + t\vec{v}) \ge J(\vec{u}) \quad \forall\vec{v} \in V;\ i.e.,$ 1 implies 3. Conversely, 3 implies the term in $J(\vec{u} + t\vec{v})$ which is linear in t must vanish, and this implies 1.■

Regularity of the Weak Solution

We have seen in the past that if $u = u(x)$ is a weak solution of the elliptic boundary value problem

$$L[u(x)] = f(x) \qquad in\ U \subset R^n$$

$$u(x) = 0 \qquad on\ \partial U$$

then

$$a(u,v) = (f,v)_0 \qquad \forall v \in V = H_0^1(U)$$

determines a unique $u \in V\ for\ each\ f \in H^{-1}(U)$. For ∂U sufficiently regular, we can show further that

$$f \in H^0(U) \;\Rightarrow\; u \in H_0^1(U) \cap H^2(U) \qquad (here\ \ \partial U \in C^2)$$

$$f \in H^m(U) \;\Rightarrow\; u \in H_0^1(U) \cap H^{m+2}(U) \qquad (here\ \ \partial U \in C^m).$$

For the Stokes system, a weak solution $\vec{u} \in V,\ p \in L^2(U)$ exists for all $\vec{f} \in L^2(U)^n$ provided ∂U is smooth. In fact, the solution possesses additional regularity, similar to that found in the elliptic problem discussed above. We can show that

$$\|\vec{u}\|_{H^2(U)^n} + \|p\|_{H^1(U)} \le C\|\vec{f}\|_{L^2(U)^n}$$

which implies

$$\vec{f} \in L^2(U)^n \;\Rightarrow\; \vec{u} \in V \cap H^2(U)^n\ and\ p \in H^1(U) \qquad (here\ \ \partial U \in C^2).$$