The Navier Stokes Equations

In preparation for considering the N-S equations, we introduce some new definitions and recall some old ones.

Vector Valued Functions of t

Let X denote a Banach space and, for $T > 0,$ let $C[0,T;X]$ denote continuous functions of t, $0 \le t \le T,$ with values in X. That is,

$$\|f(t_n) - f(t)\|_X \to 0 \quad as \quad t_n \to t \in [0,T].$$

$C[0,T;X]$ is complete for the norm $\|f\|_{C[0,T;X]} = \max_{0 \le t \le T} \|f(t)\|_X.$

For $1 \le p < \infty,$ let $L^p[0,T;X]$ denote measurable functions of t, $0 \le t \le T,$ with values in X such that

$$\|f\|_{L^p[0,T;X]} = \left(\int_0^T \|f(t)\|_X.dt\right)^{1/p} < \infty.$$

$L^p[0,T;X]$ is complete for this norm and completing the space $C[0,T;X]$ in this norm produces $L^p[0,T;X]; i.e.,\ C[0,T;X]$ is dense in $L^p[0,T;X].$

For $1 \le p < \infty,$ let $W^{1,p}[0,T;X]$ denote functions $f(t) \in L^p[0,T;X]$ such that for some $g(t)$ in $L^p[0,T;X],$

$$f(t) = f(0) + \int_0^t g(s)\,ds, \qquad t \in [0,T].$$

Then it is a classic measure theoretic result that the function f is strongly differentiable with

$$f'(t) = \lim_{h \to 0}\left[\frac{f(t+h) - f(t)}{h}\right] = g(t),$$

and since weak (distributional) and strong derivatives coincide, $W^{1,p}[0,T;X]$ consists of functions $f(t) \in L^p[0,T;X]$ such that $f'(t) \in L^p[0,T;X].$

We can define a norm on $W^{1,p}[0,T;X]$ by

$$\|f\|_{W^{1,p}[0,T;X]} = \left(\int_0^T \{\|f(t)\|_X^p. + \|f'(t)\|_X^p\}dt\right)^{1/p}$$

and $W^{1,p}[0,T;X]$ is complete in this norm and completing the space $C[0,T;X]$ in this norm produces $W^{1,p}[0,T;X]; i.e.,\ C[0,T;X]$ is dense in $W^{1,p}[0,T;X].$

Let V denote a Banach space continuously,and densely embedded in a Hilbert space H which is identified with its dual, $H = H'$. Then we have $V \subset H \subset V'$ so that the duality pairing on $V \times V'$ is realized by extending the inner product on H,

$$\langle f, v\rangle_{V \times V'} = (f, v)_H \qquad \forall v \in V.$$

Now for $1 < p < \infty$, we define

$$W_p(0,T) = \{f(t) \in L^p[0,T;V] \text{ such that } f'(t) \in L^q[0,T;V']\}$$

with

$$\|f\|_{W_p(0,T)} = \|f\|_{L^p} + \|f'\|_{L^q}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Then it is an extension of a previous result that,

i) $W_p(0,T)$ is contained in $C[0,T;H]$; *i.e.*, there exists a constant $C > 0$ such that $\forall f \in W_p(0,T)$

$$\|f\|_{C[0,T;H]} \leq C\|f\|_{W_p(0,T)}$$

ii) $\|u(\bullet)\|_H^2$ is absolutely continuous on $[0,T]$ *and*

$$\frac{d}{dt}\|u(t)\|_H^2 = 2(u'(t),u(t))_H \qquad a.e.\ t \in [0,T]$$

iii) $\forall f,g \in W_p(0,T)$ $(f(\bullet),g(\bullet))_H$ is absolutely continuous on $[0,T]$ *and*

$$\frac{d}{dt}(f(t),g(t))_H = (f'(t),g(t))_H + (f(t),g'(t))_H \qquad a.e.\ t \in [0,T]$$

iv) If V, V' are reflexive, and the embeddings $V \subset H$, $H \subset V'$ are compact and continuous, respectively, then the inclusion of $W_p(0,T)$ into $L^p[0,T;H]$ is compact.

We are going to consider the evolution N-S equations where the solutions will belong to $L^p[0,T;V]$ and the equation is understood to hold in the space $L^q[0,T;V']$. It follows from the result i) above that the solution $u(t)$ is absolutely continuous with values in H (not just in V') so the initial conditions are attained in H.

Linearized N-S Equations

In order to prepare for considering the nonlinear evolution N-S equations, we will first consider the linearized version.

$$\begin{aligned}
\partial_t\vec{u}(x,t) - \nu\nabla^2\vec{u}(x,t) + \nabla p = \vec{f}(x,t) & \quad (x,t) \in U \times (0,T) = U_T \\
div\,\vec{u}(x,t) = 0 & \quad in\ U_T \\
\vec{u} = \vec{0} & \quad (x,t) \in \partial U \times (0,T) = \partial U_T \\
\vec{u}(x,0) = \vec{u}_0(x) & \quad x \in U
\end{aligned}$$

Now recall

$K = \{\vec{\phi} \in C_C^\infty(U)^n : div\vec{\phi} = 0\} =$ divergence free vector test functions

$H =$ completion of K in the norm of $L^2(U)^n = \{\vec{u} \in L^2(U)^n : div\vec{u} = 0, T\vec{u} = 0\}$

$V =$ completion of K in the norm of $H_0^1(U)^n = \{\vec{u} \in H_0^1(U)^n : div\vec{u} = 0\}$

and

$$\forall \vec{u},\vec{v} \in H, \qquad ((\vec{u},\vec{v}))_0 = \int_U \vec{u} \bullet \vec{v}\,dx = (\vec{u},\vec{v})_H$$

$$\|\vec{u}\|_0^2 = \int_U \vec{u} \bullet \vec{u}\,dx = \|\vec{u}\|_H^2.$$

$$\forall \vec{u},\vec{v} \in V, \qquad [[\vec{u},\vec{v}]] = \sum_{i=1}^n \int_U \partial_i\vec{u} \bullet \partial_i\vec{v}\,dx = (\vec{u},\vec{v})_V$$

$$\|\vec{u}\|_V^2 = \sum_{i=1}^n \int_U \partial_i\vec{u} \bullet \partial_i\vec{u}\,dx = \|\nabla\vec{u}\|_H^2.$$

It follows from the Poincare inequality that for U bounded, $\|\vec{u}\|_V$ defines a norm on V that is equivalent to the $H^1(U)^n - norm.$ Then V, V' are reflexive, and the embeddings $V \subset H,\ H \subset V' = H^{-1}(U)^n$ are compact and continuous, respectively.

For $\vec{u} \in V,$ fixed, define $F(\vec{v}) = [[\vec{u},\vec{v}]]$ for $\vec{v} \in V$. Then $F \in V'$ and we can use the notation $F = A\vec{u}$ to indicate the dependence of F on $\vec{u}$; i.e., $A : V \to V'$. Then

$$[[\vec{u},\vec{v}]] = F(\vec{v}) = \langle A\vec{u},\vec{v}\rangle_{V\times V'} = (A\vec{u},\vec{v})_H \qquad \forall\vec{v} \in V.$$

Then we can say that $\vec{u}(t) \in L^2[0,T;V]$ is a solution of the linearized N-S system if

$$(\vec{u}'(t),\vec{v})_H + \nu[[\vec{u}(t),\vec{v}]] = (\vec{f}(t),\vec{v})_H \qquad \forall\vec{v} \in V,\ a.e. in\ (0,T).$$
$$\vec{u}(0) = \vec{u}_0$$

or, equivalently, if

$$\vec{u}'(t) + \nu A\vec{u}(t) = P\vec{f}(t), \qquad in\ V'\ a.e.\ in\ (0,T)$$
$$\vec{u}(0) = \vec{u}_0$$

where $P : L^2(U)^n = H \oplus H^\perp \to H$,

$P : H_0^1(U)^n \to V$,

and $A = -P\nabla^2 : V \to V'$.

Evidently $\vec{u}(t) \in L^2[0,T;V], \vec{u}'(t) \in L^2[0,T;V'],\ so\ \vec{u}(t) \in W_2(0,T)$ and the initial condition is satisfied in the sense of $C[0,T;H]$ for $\vec{u}_0 \in H.$

Theorem Suppose $\vec{f}(t) \in L^2[0,T;V'],\ and\ \vec{u}_0 \in H$. Then the linearized N-S system has a unique weak solution $\vec{u}(t) \in L^2[0,T;V]$, with
$\vec{u}\,'(t) \in L^2[0,T;V'],\ so\ \vec{u}(t) \in W_2(0,T)$. The initial condition is satisfied in the sense of $C[0,T;H]$.

Proof-

1) Define an approximate solution-

Let $\{\vec{w}_j\}$ denote the orthonormal basis of eigenfunctions of A and, for $N = 1,2...$

$$\vec{u}_N(t) = \sum_{j=1}^{N} c_{j,N}(t)\,\vec{w}_j,$$

where the $\{c_{j,N}\}$ are chosen such that for $1 \le k \le N$,

$$\begin{aligned}(\tfrac{d}{dt}\vec{u}_N(t),\vec{w}_k)_H + \nu[[\vec{u}_N(t),\vec{w}_k]] &= (\vec{f}(t),\vec{w}_k)_H \\ (\vec{u}_N(0),\vec{w}_k)_H &= (\vec{u}_0,\vec{w}_k)_H\end{aligned}$$

Since $[[\vec{w}_j,\vec{w}_k]] = (A\vec{w}_j,\vec{w}_k)_H = \lambda_j\,(\vec{w}_j,\vec{w}_k)_H$ this approximate system is equivalent to

$$\begin{aligned}\tfrac{d}{dt}c_{j,N}(t) + \nu\lambda_j c_{j,N}(t) &= f_j(t) \qquad 1 \le j \le N, \\ c_{j,N}(0) &= (\vec{u}_0,\vec{w}_k)_H\end{aligned}$$

and this linear system of ordinary differential equations has a unique solution for each N.

2) A-priori estimates-

For each N,

$$\left\{\begin{array}{ll} a) & \|\vec{u}_N(\bullet)\|_{C[0,T;H]} \\ b) & \|\vec{u}_N(\bullet)\|_{L^2[0,T;V]} \\ c) & \|\partial_t\vec{u}_N(\bullet)\|_{L^2[0,T;V']} \end{array}\right\} \le C\Big(\|u_0\|_H + \|\vec{f}\|_{L^2[0,T;V']}\Big)$$

3) Existence

$\{\vec{u}_N(t)\}$ is bounded in $L^2[0,T;V]$ by 2b)
$\{\partial_t\vec{u}_N(t)\}$ is bounded in $L^2[0,T;V']$ by 2c)

Then there exists a subsequence $\{\vec{u}_{N'}(t)\}$ such that

$\vec{u}_{N'}(t) \to \vec{u}(t)$ weakly in $L^2[0,T;V]$
$\partial_t\vec{u}_{N'}(t) \to \vec{v}(t)$ weakly in $L^2[0,T;V']$

Then the usual distributional argument implies $\partial_t \vec{u} = \vec{v}(t)$, hence $\vec{u}(t) \in W_2[0,T] \subset C[0,T;H]$.

It remains to be shown that the limit, $\vec{u}(t)$, is a weak solution of the linearized N-S system. Let

$$V_N = \left\{ \vec{v}(t) = \sum_{k=1}^{N} a_k(t)\vec{w}_k,\ a_k \in C^1[0,T],\ \forall k \right\}$$

Then it follows from the approximate equation that

$$\begin{array}{c} \int_0^T (\vec{u}_N'(t),\vec{v}(t))_H\,dt + \nu\int_0^T [[\vec{u}_N(t),\vec{v}(t)]]\,dt = \int_0^T (\vec{f}(t),\vec{v}(t))_H\,dt \qquad \forall \vec{v} \in V_N \\ \downarrow \qquad\qquad\qquad\qquad \downarrow \\ weakly\ in\ L^2[0,T;V'] \qquad weakly\ in\ L^2[0,T;V] \\ \downarrow \qquad\qquad\qquad\qquad \downarrow \\ \int_0^T (\vec{u}'(t),\vec{v}(t))_H\,dt + \nu\int_0^T [[\vec{u}(t),\vec{v}(t)]]\,dt = \int_0^T (\vec{f}(t),\vec{v}(t))_H\,dt \qquad \forall \vec{v} \in \bigcup_{N>0} V_N \end{array}$$

Since $\{\vec{w}_k\}$ is an orthonormal basis for V, it follows that $\bigcup_{N>0} V_N$ is dense in $L^2[0,T;V]$ and thus the limit is a weak solution for the pde. To see that $\vec{u}(0) = \vec{u}_0$, we choose a $v \in C^1[0,T;V]$ such that $\vec{v}(T) = 0$ and integrate by parts with respect to t in the approximate equation before passing to the limit.

We can even show that the approximate solution $\vec{u}_N$ converges strongly in the norm of $L^2[0,T;V]$ to $\vec{u}(t)$ using the argument we applied to linear parabolic equations in the previous course. Uniqueness is proved as it was then as well.■

The Semigroup Approach to NL Navier Stokes

Consider a bounded open set $U \subset R^n\ (n = 2,3)$ with smooth boundary ∂U. Then the nonlinear N-S system is

$$\begin{aligned} \partial_t \vec{u}(x,t) - \nu\nabla^2\vec{u}(x,t) + (\vec{u}\bullet\nabla)\vec{u} + \nabla p = \vec{f}(x,t) &\quad (x,t) \in U_T \\ div\,\vec{u}(x,t) = 0 &\quad in\ U_T \\ \vec{u} = \vec{0} &\quad (x,t) \in \partial U \times (0,T) = \partial U_T \\ \vec{u}(x,0) = \vec{u}_0(x) &\quad x \in U \end{aligned}$$

This can be expressed abstractly as

$$\begin{aligned} \vec{u}'(t) + A\vec{u}(t) = \vec{F}(\vec{u}(t),t) &\qquad in\ (0,T) \\ \vec{u}(0) = \vec{u}_0 & \end{aligned}$$

where

$$\begin{aligned} A &= -P(\nu\nabla^2):D_A \to H \\ \vec{F}(\vec{u}(t),t) &= P\vec{f}(\bullet,t) - P(\vec{u}\bullet\nabla)\vec{u}(t). \end{aligned}$$

We must be able to show that

$$\vec{F} : H_\alpha \times [0,\infty) \to H$$

such that for some $0 < \mu < 1$,

$$\|\vec{F}(\vec{u}(t),t) - \vec{F}(\vec{v}(\tau),\tau)\|_H \le C_R\left(|t-\tau|^\mu + \|\vec{u}-\vec{v}\|_\alpha\right) \qquad \forall t,\tau \in [0,T],\ \vec{u},\vec{v} \in B_R(0) \subset H_\alpha$$

If $\vec{f}(t)$ is Lipschitz continuous with values in H, then

$$\left|\left|P\vec{f}(t) - P\vec{f}(\tau)\right|\right|_H \le C|t-\tau| \qquad \forall t,\tau \in [0,\infty).$$

Now, $H_\alpha \subset W^{1,2}(U)\ \ if\ \ 2\alpha > 1\ \ and\ \ H_\alpha \subset C^0(\bar{U})\ \ if\ \ 4\alpha > n$. Then for $N(\vec{u}) = (\vec{u} \bullet \nabla)\vec{u}$ we have

$$\|N(\vec{u})\|_H = \left(\int_U |(\vec{u}\bullet\nabla)\vec{u}|^2 dx\right)^{1/2} \le \|\vec{u}\|_\infty \|\nabla\vec{u}\|_0$$

and

$$\|\vec{u}\|_\infty \le C_1\|\vec{u}\|_\alpha \qquad if\ \ 4\alpha > n$$
$$\|\nabla\vec{u}\|_0 = \|\vec{u}\|_V \le C_2\|\vec{u}\|_\alpha, \qquad if\ 2\alpha > 1.$$

Therefore, $$\|N(\vec{u})\|_H \le C\|\vec{u}\|_\alpha^2 \qquad for \left\{\begin{array}{ll} \alpha > \frac{1}{2} & n = 2 \\ \alpha > \frac{3}{4} & n = 3 \end{array}\right\}$$

which implies $\quad N : H_\alpha \to H\ \ is\ bounded\ for\ \ \alpha > \frac{n}{4}.$

In addition,

$$\|N(\vec{u}) - N(\vec{v})\|_H = \|\vec{u}\bullet\nabla\vec{u} - \vec{v}\bullet\nabla\vec{v}\|_H$$

$$\le \|(\vec{u}\bullet\nabla)(\vec{u}-\vec{v})\|_H + \|\vec{u}-\vec{v}\|_\infty\|\nabla\vec{v}\|_H$$

$$\le \|\vec{u}\|_\alpha\|(\vec{u}-\vec{v})\|_V + \|\vec{u}-\vec{v}\|_\alpha\|\vec{v}\|_V$$

$$\le \left(\|\vec{u}\|_\alpha + \|\vec{v}\|_\alpha\right)\|\vec{u}-\vec{v}\|_\alpha \le 2R\,\|\vec{u}-\vec{v}\|_\alpha$$

for all $\vec{u},\vec{v} \in B_R(0) \subset H_\alpha\ provided\ \alpha > \frac{3}{4}$. (when $n = 2,\ \alpha > \frac{1}{2}\ \ is$ sufficient). Then we have,

Theorem For $\alpha > \frac{n}{4}$ and all $\vec{u}_0 \in H_\alpha$, and for $\vec{f}(t)$ Lipschitz continuous in t with values in H, there is a positive $T > 0$ and $\vec{u} \in C[0,T;H_\alpha] \cap C^1[0,T;H_0]$ such that u solves the abstract N-S initial value problem (in the stong sense since $-A$ generates an analytic semigroup). In addition,

$$\nabla p = \vec{f} - \partial_t \vec{u} - A\vec{u} - \vec{u} \bullet \nabla \vec{u} \in C[0,T;H_0].$$

The solution is unique but there is no information about the size of T. We could try to establish that

$$\|\vec{u}(t)\|_\alpha \leq K \qquad \textit{for all } t$$

by means of an energy estimate; i.e. let

$$E(t) = \int_U \left\{ \tfrac{1}{2} |\nabla \vec{u}(t)|^2 - \Phi(\vec{u}) \right\} dx \qquad \Phi'(\vec{u}) = F(\vec{u}).$$

Then under appropriate hypotheses on F, we may be able to show that $E'(t) \leq 0,$ *and*

$$\|\vec{u}(t)\|_{1/2}^2 \leq k_0 E(t) + k_1.$$

This gives a uniform bound on $\|\vec{u}(t)\|_{1/2}$ but since we need $\alpha > \frac{1}{2}$ to get existence of a solution in H_α, this approach fails. (i.e., energy estimates always involve the so-called "energy norm", $\|\vec{u}(t)\|_{1/2} = \|\vec{u}(t)\|_V$).