Applications of Semigroups to Nonlinear IVP’s

1. The Abstract IVP
Consider the following nonlinear initial value problem

\[ u'(t) + Au(t) = F(u(t)) \quad 0 < t < T, \quad u(0) = u_0 \]  

(1.1)

where \(-A : D_A \rightarrow H\) generates a \(C^0 - s\) of contractions on \(H\). Of course this includes the special case that the semigroup generated by \(-A\) is analytic. A strong solution of (1) on \([0, T]\) is a function \(u(t) \in C^0([0, T] : H) \cap C^1((0, T) : H)\) which solves the equation and we will define a function \(u(t)\) to be a mild solution of (1) if \(u(t) \in C^0([0, T] : H)\) satisfies

\[ u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds \quad 0 < t < T. \]  

(1.2)

The simplest existence proofs for problems like this make the assumption that \(F : V \rightarrow H\) is locally Lipschitz; i.e., \(V\) denotes a closed subspace of \(H\) (\(V = H\) is allowed) and for some \(R > 0\), there exists \(C_R > 0\) such that

\[ ||F(u) - F(v)||_H \leq C_R||u - v||_V \quad \forall u, v \in B_R(0) \subset V \]  

(1.3)

For some nonlinearities it will suffice to take \(V = H\), while for others it will be necessary to choose \(V\) to be an appropriate proper closed subspace of \(H\). In these cases we will suppose that \(S(t)\) maps \(H\) into \(V \subset D_\phi\) with \(||S(t)x||_V \leq CV||x||_H\), and, for convenience we will assume \(C_V = 1\).

To show that (1) has a mild solution under the assumption (3), let

\[ \Phi(u) = \int_0^t S(t-s)F(u(s))ds \quad \text{and} \quad v(t) = S(t)u_0, \]

also

\[ R = 2||u_0||_V \quad \text{and} \quad K_R = RC_R + ||F(0)||_H. \]

Then

\[ ||F(u)||_H - ||F(0)||_H \leq ||F(u) - F(0)||_H \leq C_R||u||_V \leq RC_R \]

and

\[ ||F(u)||_H \leq K_R \quad \forall u \in B_R(0) \subset V. \]

This bound on \(||F(u)||_H\) implies

\[ ||\Phi(u(t))||_V \leq \max_{0 \leq s \leq T} ||S(t-s)F(u(s))||_V \leq TK_R \]

if \(u(t) \in B_R(0)\) for \(0 \leq t \leq T\). Now if we let
\[ M_R = \{ u \in C([0, T]; H) : \|u(t)\|_V \leq R, \ 0 \leq t \leq T \} \]

Then for \( u \in M_R \) and \( 0 < T < R/(2K_R) \) we have
\[
\|\Phi(u(t))\|_V \leq TK_R < R/2 = \|u_0\|_V
\]
i.e.,
\[
\Phi : M_R \rightarrow M_R \quad \text{for} \quad 0 \leq t \leq T < \frac{R}{2K_R}
\]

In addition, for \( 0 \leq t \leq T \),
\[
\|\Phi(u(t)) - \Phi(w(t))\|_H \leq C_R \|u(t) - w(t)\|_V \quad \forall u, v \in M_R
\]
hence, for \( t < 1/C_R \), \( \Phi \) is a strict contraction on \( M_R \). Now let
\[
T_0 = \min[1/C_R, R/2K_R]
\]
Then for \( u \in M_R \) and \( 0 \leq t \leq T_0 \),
\[
\|v(t) + \Phi(u(t))\|_V \leq \|u_0\|_V + \|\Phi(u(t))\|_V \leq 2\|u_0\|_V = R
\]
and it follows that \( M_R \ni u \mapsto v + \Phi(u) \in M_R \) is a strict contraction. Then there is a unique fixed point, \( \hat{u} \in M_R \) such that
\[
\hat{u}(t) = v(t) + \Phi(\hat{u}(t)), \quad 0 \leq t \leq T_0
\]
i.e., \( \hat{u} \) is a mild solution of the IVP. In order to prove that \( \hat{u} \) is, in fact, a strong solution to the IVP, additional hypotheses on \( A \) or on \( F \) are needed. For example, if \( A \) generates an analytic semigroup, then \( \hat{u} \) would have the additional smoothness required of a strong solution. Also if additional smoothness on \( F \) were assumed, we may be able to show the mild solution is strong.

Since the solution has only been shown to exist for \( 0 \leq t \leq T_0 \), it is referred to as a local solution. In an effort to extend the solution to larger time, suppose we use \( u_1 = \hat{u}(T_0) \) as the initial condition for a new IVP and follow the same procedure to obtain a new mild solution on an interval \( [T_0, T_1] \) for some \( T_1 > T_0 \). Repeating this procedure \( N \) times leads to solutions on \( [0, T_0] \cup [T_0, T_1] \cup [T_1, T_2] \cup \ldots \cup [T_{N-1}, T_N] = [0, T_N] \). In general, the length \( |[T_j, T_{j+1}]| \) tends to zero with increasing \( j \) due to the fact that \( R, C_R, K_R \) grow as \( T \) increases. However, if it is known, say from some a-priori estimate of the solution, that any solution of the IVP must satisfy \( \|u(t)\|_V \leq C \) for \( 0 \leq t \leq T \), then we may take \( R = \max[2\|u_0\|_H, C] \) in the procedure just described. Then we can divide \([0,T]\) into subintervals \([T_j, T_{j+1}]\) of uniform length and in this way, obtain a solution for the interval \([0,T]\); i.e., a uniform bound on solutions implies a global solution.

The nonlinear operator \( \Xi[u(t)] = v(t) + \Phi(u(t)) : H \rightarrow H \) may be interpreted as the continuous flow on \( H \) associated with the IVP.
2. A Nonlinear Diffusion Equation on \( \mathbb{R}^n \)

Consider the problem

\[
\begin{align*}
\partial_t u(x,t) &= \nabla^2 u(x,t) + f(u(x,t)) & x \in \mathbb{R}^n, & t > 0 \\
u(x,0) &= u_0(x) & x \in \mathbb{R}^n.
\end{align*}
\] (2.1)

In this problem we take, instead of a Hilbert space \( H \), the Banach space of functions which are defined and continuous on \( \mathbb{R}^n \) and have a finite max. This linear space of functions \( X = C_b(\mathbb{R}^n) \) is a Banach space for the sup norm. We assume also that the nonlinearity, \( f : \mathbb{R} \to \mathbb{R} \) satisfies,

\[
|f(u) - f(v)| \leq C_R |u - v| \quad \forall |u|,|v| \leq R
\] (2.2)

Note that \( f(u) = u^2 \) satisfies condition (2.2) for \( C_R = 2R \). Then (2.2) implies that \( F(u) = f(u(x,t)) \) satisfies the condition (1.3) with \( H = V = X \), and, since the composition of continuous functions is continuous, that \( F(u) = f(u(x,t)) \) maps \( X \) to itself.

Since the operator \( A = -\nabla^2 \) on \( D_A = \{ u \in X : Au \in X \} = C^2(\mathbb{R}^n) \cap C_b(\mathbb{R}^n) \) can be shown to generate a \( C^0 \) semigroup of contractions on \( X \), it follows from the result of the previous section that the initial value problem has a unique mild solution, \( \hat{u}(x,t) \) which satisfies,

\[
\hat{u}(t) = S(t)u_0 + \int_0^t S(t-s)F(\hat{u}(s))ds \quad 0 < t < T_0
\]
i.e.,

\[
\hat{u}(x,t) = \int_{\mathbb{R}^n} K(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)f(\hat{u}(y,s))dyds.
\] (2.3)

where

\[
K(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0.
\]

Since the semigroup generated by \( -A = \nabla^2 \) is, in fact, analytic, we can show that the mild solution to the IVP is actually a strong solution. This follows from the fact that when the semigroup is analytic, the abstract IVP has a strong solution when the inhomogeneous term \( f(t) \) is only Lipschitz continuous in \( t \). The condition (2.2) is sufficient to imply that \( f(t) = f(u(x,t)) \) is Lipschitz in \( t \) for any \( u(x,t) \in X \).

In addition, for this problem it is possible to use monotonicity methods to establish uniform bounds on the solution under appropriate conditions on \( f \). When \( f \) is such that such bounds can be established, the solution can be shown to be global in \( t \).

3. An IBVP in 1-dimension

Consider the problem

\[
\begin{align*}
\partial_t u(x,t) - \partial_x x u(x,t) &= f(u(x,t)) & 0 < x < 1, & t > 0 \\
u(x,0) &= u_0(x) & 0 < x < 1, \\
u(0,t) &= u(1,t) = 0 & t > 0.
\end{align*}
\]
where we suppose \( f \in C^1(R) \).
Let \( H = L^2(0,1) \) and \( V = H^1_0(0,1) \). Then we can show that

\[
V \subset C^{0,a}(0,1) \quad \text{for} \quad 0 < a \leq 1/2.
\]

i.e., for \( u \in V \), and \( 0 \leq x, y \leq 1 \),

\[
|u(x) - u(y)| = \left| \int_y^x u'(s) \, ds \right| \leq \left( \int_y^x 1^2 \, ds \right)^{1/2} \left( \int_y^x u'(s)^2 \, ds \right)^{1/2}
\]

\[
\leq |x - y|^{1/2} \left( \int_0^1 u'(s)^2 \, ds \right)^{1/2} \leq \| u \|_V |x - y|^{1/2}
\]

Then it follows that for \( 0 \leq x \leq 1 \), \( |u(x)| \leq \| u \|_V \); i.e., \( \| u \|_\infty \leq \| u \|_V \). In particular then for \( u \in V \), \( f(u) \in H \) so \( F = f(u) \) maps \( V \) to \( H \). Now, for \( u, v \in B_R(0) \subset V \),

\[
\| f(u) - f(v) \|_H^2 = \int_0^1 |f(u(x)) - f(v(x))|^2 \, dx
\]

\[
\leq (\max_{|s| \leq R} |f'(s)|)^2 \int_0^1 |u(x) - v(x)|^2 \, dx
\]

\[
\leq C_R \| u - v \|_V^2 \leq C_R \| u - v \|_V^2
\]

and we see that \( f : V \mapsto H \) is locally Lipschitz. It follows from the results of section 1 that
the abstract IVP has a unique mild solution, \( u \in C([0,T]; H) \) for \( T > 0 \), sufficiently small.
However, since the semigroup generated by \(-A\) is, in fact, an analytic semigroup, the
Lipschitz smoothness of \( f \) is sufficient to imply that the mild solution is actually strong.

Note that we used that \( V \subset C^{0,1/2}([0,1]) \subset H \) in order to assert that \( f(u) \in H \) for
\( u \in V \) and that

\( u, v \in B_R(0) \subset V \) implies \( \| u \|_\infty \leq R \), and \( \| v \|_\infty \leq R \)

which leads then to the result, \( |f(u) - f(v)| \leq \max_{|s| \leq R} |f'(s)| |u - v| \); i.e., this is a case where we
have to take \( V \) to be an appropriate closed subspace of \( H \) in order to get the behavior we
need for \( f \).

4. A Semilinear IBVP on \( R^1 \)
Consider the semilinear problem

\[
\partial_t u(x,t) - \partial_{xx} u(x,t) + u(x,t)\partial_x u(x,t) = f(u(x,t)) \quad 0 < x < 1, \ t > 0
\]

\[
\begin{align*}
  u(x,0) &= u_0(x) & 0 < x < 1, & \quad (4.1) \\
  u(0,t) &= u(1,t) = 0 & t > 0, \\
\end{align*}
\]

where we suppose \( f \in C^1(R) \). Let
\[ F(u) = f(u) - u \partial_x u \]

\[ H = L^2(0,1) \quad V = H_0^1(0,1) \subset C^{0,1/2}([0,1]) \]

Then \( f : V \mapsto H \)

and \( \|u \partial_x u\|_H \leq \|u\|_V \|\partial_x u\|_H \leq \|u\|_V^2 \)

so we have \( F : V \mapsto H \). Moreover, for all \( u, v \in B_R(0) \subset V \),

\[
\|u \partial_x u - v \partial_x v\|_H \leq \|u(\partial_x u - \partial_x v)\|_H + \|(u - v)\partial_x v\|_H \leq \|u\|_V \|u - v\|_V + \|(u - v)\|_V \|v\|_V \leq 2R \|u - v\|_V
\]

and this implies \( F \) is locally Lipschitz on \( V \). It follows then that the abstract IVP has a unique mild solution which can again be seen to be a strong solution due to the fact that \( -A \) generates an analytic semigroup on \( H \). The strong solution is only local in \( t \) unless some a-priori bound on the solution can be established.

5. A Semilinear IBVP on \( \mathbb{R}^n \), \( n=2,3 \)

The previous two examples were set in one space dimension where it happens that \( V \subset C^{0,\alpha}(0,1) \) for \( 0 < \alpha \leq 1/2 \). For \( n \geq 2 \), the Sobolev embedding theorem changes the situation and we have to deal more carefully with the function spaces in order to get the Lipschitz behavior for the nonlinearity.

For \( U \) a bounded open set in \( R^n \), \( n \geq 2 \) and for \( \alpha \geq 0 \), define

\[ H_\alpha(U) = \left\{ u \in H^\alpha(U) : \sum_{j \geq 1} |\lambda_j|^{2\alpha} |(u, \varphi_j)_H|^2 < \infty \right\} \]

where \( \{\varphi_j\}_{j \geq 1} \) denote the orthonormal family of eigenfunctions for \( A = -\nabla^2 \) on \( V = H_0^1(U) \); i.e.,

\[ H = H^0(U) \ni u = \sum_{j \geq 1} (u, \varphi_j)_H \varphi_j \quad \|u\|_H^2 = \sum_{j \geq 1} |(u, \varphi_j)_H|^2 \]

\[ H_1 = D_A = \left\{ u \in H : Au = \sum_{j \geq 1} \lambda_j (u, \varphi_j)_H \varphi_j \in H \right\} \]

i.e., \( u \in D_A \) iff \( \|Au\|_H^2 = \sum_{j \geq 1} |\lambda_j|^2 |(u, \varphi_j)_H|^2 < \infty \)

for \( u \in H_\alpha, \quad A^\alpha u = \sum_{j \geq 1} \lambda_j^\alpha (u, \varphi_j)_H \varphi_j \quad 0 \leq \alpha \leq 1 \),
\[ \|u\|_{\alpha}^2 = \|A^\alpha u\|_H^2 = \sum_{j \geq 1} |\lambda_j|^{2\alpha} |(u, \varphi_j)_H|^2 \]

This defines a sequence of linear spaces,

\[ D_A = H_1 \subset H_\alpha \subset H_0 = H^0(U), \quad 0 < \alpha < 1. \]

Evidently, \( H_\alpha \) is a Hilbert space for

\[ (u, v)_\alpha = (u, v)_H + (A^{\alpha/2}u, A^{\alpha/2}v)_H = \sum_{j \geq 1} (1 + |\lambda_j|^{2\alpha}) |u_jv_j| \]

i.e.,

\[ \|u\|_{\alpha}^2 = \|u\|_H^2 + \|A^\alpha u\|_H^2 \]

And since this can be seen to be the graph norm on \( D_A \), it follows from the closed graph theorem that \( H_\alpha \) is a Banach space for this norm. Of course the norm then supports this inner product and \( H_\alpha \) becomes a Hilbert space. In particular, \( H_{1/2} = H_0^0(U) \).

**Embedding Results**

We state now some results regarding the embedding of the \( H_\alpha \) spaces.

If \( H_0^0(U) \cap H^2(U) \subset D_A = H_1 \subset H_\alpha \subset H_0 = H^0(U), \quad 0 < \alpha < 1. \)

then we can show that

\( H_\alpha \) is continuously embedded in \( W^{\alpha,4}(U) \) if \( \begin{cases} 
2\alpha > p \\
2\alpha - n/2 > p - n/q 
\end{cases} \)

\( H_\alpha \) is continuously embedded in \( C^m(\bar{U}) \) if \( 2\alpha - n/2 > m \)

Now consider

\[ \begin{align*}
\partial_t u(x, t) - \nabla^2 u(x, t) &= f(u(x, t)) & x \in U \subset \mathbb{R}^n, & t > 0 \\
u(x, 0) &= u_0(x) & x \in U \\
u(x, t) &= 0 & x \in \Gamma, & t > 0,
\end{align*} \]

where \( f \in C^1(\mathbb{R}). \) Then \( F(u) = f(u(x, t)) : H_\alpha \to H \) provided \( H_\alpha \hookrightarrow C^0(U) \); i.e., for \( \alpha > n/4. \)

In addition, \( F \) is locally Lipschitz if

\[ u, v \in B_R(0) \subset H_\alpha \implies \|u\|_{\alpha}, \|v\|_{\alpha} \leq R \]

Again, we need \( H_\alpha \) continuously embedded in \( C^0(\bar{U}) \) which means that \( \alpha > n/4. \) It follows that for \( u_0 \in H_\alpha \) with \( \alpha > n/4 \) there is a unique mild solution for the IBVP, \( \tilde{u}(t) \in C([0, T]; H) \) for sufficiently small \( T > 0. \) Since the semigroup generated by \( -A \) is analytic here, the solution is actually a strong solution belonging to \( C^0([0, T]; H) \cap C^1((0, T) : H) \). Note that for \( n \geq 2 \) it is not sufficient to choose \( H_{1/2} = H_0^0(U) \) as the closed subspace of \( H \) which leads
to Lipschitz behavior for $F$.

Now let us consider the IBVP in the more difficult case where $n = 3$ and the nonlinearity $F(u) = f(u(x, t))$ is given by

$$f(u) = \sum_{j=1}^{3} u(\partial u/\partial x_j)$$

This nonlinearity is more difficult to deal with than the previous $f \in C^1(R)$ and we need some lemmas before trying to prove existence of the solution to the IBVP.

**Lemma 1** There exists a constant $C > 0$, such that for all $u \in H_1 = D_A$,

Lemma $|u(x) - u(y)| \leq C \|Au\|_H |x - y|^{1/2} \forall x, y \in R^3$

Proof- For $\phi \in C^0_0(U)$ we have the classical representation for a solution of Poisson's equation in terms of a fundamental solution, (cf sec 2.2.1 in the Evans text)

$$\phi(x) = C \int_U \frac{\nabla^2 \phi(y)}{|x-y|} dy$$

for $C$ an appropriate constant. Applying the C-S inequality to this expression leads to

$$|\phi(x) - \phi(z)|^2 \leq C^2 \left( \int_U \frac{\nabla^2 \phi(y)}{|x-y|} \left\{ \frac{1}{|x-y|} - \frac{1}{|z-y|} \right\} dy \right)^2$$

$$\leq C^2 \int_U |\nabla^2 \phi(y)|^2 dy \cdot \int_U \left\{ \frac{1}{|x-y|} - \frac{1}{|z-y|} \right\}^2 dy$$

But

$$\int_U \left\{ \frac{1}{|x-y|} - \frac{1}{|z-y|} \right\}^2 dy \leq C_U |x-z|$$

for $C_U > 0$ depending only on U. Then it follows that

$$|\phi(x) - \phi(z)| \leq C |\|A\phi\|_H|x-z|^{1/2}$$

Since $C^0_0(U)$ is dense in $D_A = H_1 \subset C^0(U)$, we can approximate any $u \in D_A$ by $\{\phi_n\} \subset C^0_0(U)$ and pass to the limit to get the result.

**Lemma 2** There exists a constant $C > 0$, such that for all $u \in H_1 = D_A$,

Lemma $\|u\|_4^4 \leq C \|Au\|_H^3 \|u\|_H$

Proof- The embedding results imply $D_A = H_1 \subset C^0(U)$ and, assuming the boundary $\Gamma$ is smooth, we have that $u|\Gamma = 0$, since $H^0_0(U) \cap H^2(U)$ is dense in $D_A = H_1$. Now if $u$ is identically zero, the result is trivial so suppose $\|u\|_\infty = \text{ess sup}_U |u(x)| = L > 0$.

We have from the previous lemma

$$|u(x) - u(y)| \leq K |x-y|^{1/2} \text{ for } K = C \|Au\|_H$$
and WOLG we may suppose \( L = |u(0)| \). Let \( R = (L/K)^2 \) and consider the open ball, \( B_R(0) \subset R^1 \). For \( x \in B_R(0) \)

\[
|u(x)| > |u(0)| - |u(0) - u(x)| \geq L - K|x|^{1/2} > L - (K/L) = 0
\]

Since \( u|_{\Gamma} = 0 \) this last estimate implies \( B_R(0) \subset U \) and for \( x \in B_R(0) \), \( |u(x)| \geq L - K|x|^{1/2} \). Now the result follows from,

\[
\|u\|^2_H \geq \int_{B_R(0)} |u(x)|^2 \, dx \geq \int_{B_R(0)} \left| L - K|x|^{1/2} \right|^2 \, dx
\]

\[
\geq 4\pi L^2 R^3 \int_0^1 (1 - z^{1/2})^2 z^2 \, dz = CL^2 R^3 = CL^2 K^{-6}
\]

i.e.,

\[
L^4 \leq CK^3 \|u\|_H.
\]

**Lemma 3** For \( 1 \geq \alpha > 3/4 \), and \( \forall u, v \in D_\alpha \)

1) \( \hat{f}: H_\alpha \rightarrow H, \) with \( \|f(u)\|_H \leq C\|A^\alpha u\|_H\|A^{1/2}u\|_H \)

2) \( \|f(u) - f(v)\|_H \leq C(\|A^\alpha u\|_H\|A^{1/2}u - A^{1/2}v\|_H + \|A^{1/2}v\|_H\|A^\alpha u - A^\alpha v\|_H) \)

**Proof-** Note that the embedding result asserts that for \( 1 \geq \alpha > 3/4 \), \( H_\alpha \) is continuously embedded in \( C(\overline{U}) \). This implies that there exists a constant \( C > 0 \), depending on \( U \) and \( \alpha \) such that for all \( u \in D_\alpha, \|u\|_\alpha \leq C\|A^\alpha u\|_H \). Then for \( u \in D_\alpha, \ u \in L^\infty(U) \) and \( \partial u / \partial x_i \in L^2(U) = H \) so \( f(u) \in H \). Moreover

\[
\|f(u)\|_H \leq \|u\|_\alpha \|\nabla u\|_H \leq C\|A^\alpha u\|_H\|\nabla u\|_H \leq C\|A^\alpha u\|_H\|A^{1/2}u\|_H.
\]

This proves 1). Now note that

\[
\|f(u) - f(v)\|_H \leq \|u\|_\alpha \|\nabla u - \nabla v\|_H = \|u\|_\alpha \|u\|_H \|\nabla (u - v) - (u - v)\|_H
\]

\[
\leq \|u\|_\alpha \|\nabla (u - v)\|_H + \|u - v\|_\alpha \|\nabla v\|_H
\]

\[
\leq C(\|A^\alpha u\|_H\|A^{1/2}u - A^{1/2}v\|_H + \|A^{1/2}v\|_H\|A^\alpha u - A^\alpha v\|_H).
\]

This proves 2). \[\]

Now we can show the results needed to establish existence for the solution of the IBVP. Since \( D_\alpha = H_1 \subset H_\alpha \subset H_0 = H^0(U) \), \( 0 < \alpha < 1 \), it follows that the mapping \( f \) can be extended from \( H_1 \) to \( H_\alpha \) for \( 1 \geq \alpha > 3/4 \). Moreover, \( H_{3/4} \subset H_{1/2} \) and

\[
\|A^{1/2}u - A^{1/2}v\|_H \leq \|A^{3/4}u - A^{3/4}v\|_H
\]

It follows that \( f \) satisfies, for \( 1 \geq \alpha > 3/4 \),

\[
\|f(u) - f(v)\|_H \leq C(\|A^\alpha u\|_H \|A^\alpha v\|_H \|A^\alpha u - A^\alpha v\|_H)
\]
i.e., \( f : H_a \to H \) is locally Lipschitz for \( 1 \geq \alpha > 3/4 \). Then the IBVP has a unique mild solution for every \( u_0 \in H_a, \ 1 \geq \alpha > 3/4 \). Since the semigroup \( S(t) \), generated by \( -A \) is analytic, this is also a strong solution.