Evolution Navier Stokes Equations

For $U \subset R^n,\ n \le 4,$ a bounded open set with a smooth boundary, Γ, consider the full Navier Stokes system for $f \in L^2(U_T)\ and\ u_0 \in L^2(U),$

$$\begin{aligned} \partial_t\vec{u}(x,t) - \nu\nabla^2\vec{u}(x,t) + (\vec{u}\bullet\nabla)\vec{u} + \nabla p = \vec{f} \quad & in\ U_T \\ div\ \vec{u} = 0 \quad & in\ U_T \\ \vec{u}(x,t) = 0 \quad & on\ \Gamma_T \\ \vec{u}(x,0) = \vec{u}_0 \quad & in\ U \end{aligned}$$

We define a weak solution for this system to consist of a vector valued function $\vec{u}(x,t) \in L^2[0,T : V]$ and a scalar valued $p(x,t) \in L^2[0,T : H]$ such that for all $\vec{v} \in V$ and almost all $t \in (0,T),$

$$\begin{aligned} (\vec{u}'(t),\vec{v})_H + \nu[[\vec{u}(t),\vec{v}]] + b(\vec{u}(t),\vec{u}(t),\vec{v}) &= (\vec{f}(t),\vec{v})_H \\ (\vec{u}(0) - \vec{u}_0,\vec{v})_H &= 0 \end{aligned}$$

and

$$\nabla p(t) = \vec{f}(t) - \nu A\vec{u}(t) - B[\vec{u}(t)] - \vec{u}'(t)$$

where $A, B[\bullet]$ are defined by

$$F_{\vec{u}}(\vec{v}) = [[\vec{u}(t),\vec{v}]] = \langle A\vec{u}(t),\vec{v}\rangle_{V'\times V} = (A\vec{u}(t),\vec{v})_H \qquad \forall\vec{v} \in V$$

$$\langle B[\vec{u}(t)],\vec{v}\rangle_{V'\times V} = b(\vec{u}(t),\vec{u}(t),\vec{v}) \qquad \forall\vec{v} \in V$$

With respect to the last definition, recall that

$$b(\vec{u},\vec{v},\vec{w}) = \sum_{i,j=1}^n \int_U u_i\partial_i v_j w_j\,dx \qquad \forall\vec{u},\vec{v},\vec{w} \in H_o^1(U)^n$$

Since the extended Holder inequality implies

$$\left|\int_U u_i\partial_i v_j w_j\,dx\right| \le |u_i|_4|\partial_i v_j|_2|w_j|_4$$

and $H_o^1(U)^n$ is continuously embedded in $L^4(U)^n$ when $n \le 4,$ it follows that

$$|\int_U u_i\partial_i v_j w_j\,dx| \le C\|u_i\|_{H_o^1(U)}\|v_j\|_{H_o^1(U)}\|w_j\|_{H_o^1(U)}.$$

This gives,

$$|b(\vec{u}(t),\vec{u}(t),\vec{v})| \le C\|\vec{u}\|_V^2\ \|\vec{v}\|_V$$

and so

$$\langle B[\vec{u}(t)],\vec{v}\rangle_{V'\times V} = b(\vec{u}(t),\vec{u}(t),\vec{v}) \qquad \forall\vec{v}\in V$$

defines an element $B[\vec{u}(t)]$ of V'. Then the weak equation can be written as

$$\vec{u}'(t) + \nu A\vec{u}(t) + B[\vec{u}(t)] = P\vec{f}(t), \qquad a.e.\ in\ (0,T)$$
$$\vec{u}(0) = P\vec{u}_o$$

We would now like to conclude from $\vec{u}'(t) = P\vec{f}(t) - \nu A\vec{u}(t) - B[\vec{u}(t)]$ that $\vec{u}'(t) \in L^2[0,T : V']$ but we do not, at this point, know to what space $B[\vec{u}(t)]$ belongs. Here, P denotes the projection from $L^2[U]$ to H. From this point on, we do not distinguish between $P\vec{f}(t)$ *and* $\vec{f}(t)$, *nor* $P\vec{u}_o$ *and* $\vec{u}_o$.

Lemma 1 If $\vec{u}(t) \in L^2[0,T : V]$ then $B[\vec{u}(t)] \in L^1[0,T : V']$

Proof- For $\vec{u}(t) \in L^2[0,T : V]$,

$$\int_0^T \|B[\vec{u}(t)]\|_{V'}dt = \int_0^T \sup_{\|v\|_V=1} |\langle B[\vec{u}(t)],v\rangle| dt = \int_0^T \sup_{\|v\|_V=1} |b(\vec{u}(t),\vec{u}(t),\vec{v})|\,dt$$

$$\leq C\int_0^T \|\vec{u}(t)\|_V^2\,dt = C\|u\|^2_{L^2[0,T:V]} < \infty$$

This proves that $B[\vec{u}(t)] \in L^1[0,T : V']$.

As a result of the lemma, we have that if $\vec{u}(t) \in L^2[0,T : V]$ solves the weak NS equation, then $\vec{u}'(t) \in L^1[0,T : V']$. Since $L^2[0,T : V]$ is continuously embedded in $L^1[0,T : V']$ we have $\vec{u}(t) \in W^{1,1}[0,T : V']$ and it follows then from a previous lemma that $\vec{u}(t) \approx \tilde{u}(t) \in C[0,T : V']$ (not $C[0,T : H]$!). We can improve on this when $n \leq 4$.

Lemma 2 Suppose $\vec{u}(t) \in L^2[0,T : V]$ solves the weak NS system with $n \leq 4$. Then

$$\vec{u}(t) \in L^{2p}[0,T : L^4(U)^n] \quad and \qquad \vec{u}'(t) \in L^p[0,T : V'] \qquad p \leq 4/n.$$

In particular, we have the following cases when $n = 2,3,4$:

(i) $n = 2$ $\quad \vec{u}(t) \in L^4[0,T : L^4(U)^n] \simeq L^4(U_T)^n$
and $\quad \vec{u}'(t) \in L^2[0,T : V']$ $\quad$ (*then* $\vec{u}(t) \approx \tilde{u}(t) \in C[0,T : H]$)

(ii) $n = 3$ $\quad \vec{u}(t) \in L^{8/3}[0,T : L^4(U)^n]$
$\vec{u}'(t) \in L^{4/3}[0,T : V']$

(iii) $n = 4$ **Lemma 2 = Lemma 1**

Proof- If $\vec{u}(t) \in L^2[0,T : V]$ solves the weak NS equation, then

$$\underbrace{(\vec{u}'(t),\vec{u}(t))_H} + \nu[[\vec{u}(t),\vec{u}(t)]] + b(\vec{u}(t),\vec{u}(t),\vec{u}(t)) = (\vec{f}(t),\vec{u}(t))_H$$

$$1/2 \quad d/dt \ \|\vec{u}(t)\|_H^2 + \nu\|\vec{u}(t)\|_V^2 \leq \|f\|_{V'}\|\vec{u}(t)\|_V \quad \left(recall \quad b(\vec{u}(t),\vec{u}(t),\vec{u}(t)) = 0 \right)$$

Then

$$1/2 \quad d/dt \ \|\vec{u}(t)\|_H^2 + \nu\|\vec{u}(t)\|_V^2 \leq 1/2\left[1/\nu\|f(t)\|_{V'}^2 + \nu\|\vec{u}(t)\|_V^2\right]$$

and

$$d/dt \ \|\vec{u}(t)\|_H^2 + \nu\|\vec{u}(t)\|_V^2 \leq 1/\nu\|f(t)\|_{V'}^2 \qquad (1)$$

(a) *First energy estimate*- Integrate (1) from $t = 0 \ to \ t = \tau \leq T$,

$$\|\vec{u}(\tau)\|_H^2 \leq \|\vec{u}_0\|_H^2 + 1/\nu\|f\|_{L^2[0T:V']}^2$$

Since this holds for all $\tau \leq T$,

$$\max_{0\leq\tau\leq T} \|\vec{u}(\tau)\|_H^2 = \|\vec{u}(\bullet)\|_{C[0,T:H]}^2 \leq \|\vec{u}_0\|_H^2 + 1/\nu\|f\|_{L^2[0T:V']}^2$$

(b) *Second energy estimate*- Integrate (1) from $t = 0 \ to \ t = T$,

$$\nu\int_0^T \|\vec{u}(t)\|_V^2 \, dt \leq \|\vec{u}_0\|_H^2 + 1/\nu\|f\|_{L^2[0T:V']}^2$$

i.e.,

$$\|\vec{u}(\bullet)\|_{L^2[0,T:V]}^2 \leq 1/\nu \, \|\vec{u}_0\|_H^2 + 1/\nu^2 \, \|f\|_{L^2[0T:V']}^2$$

Now recall,

$$\|\vec{u}\|_q \leq C\|\nabla\vec{u}\|_p^\lambda \, \|\vec{u}\|_r^{1-\lambda} \qquad \forall u \in C_0^\infty(U)$$

where

$$0 \leq \lambda \leq 1, \qquad 1/q = \lambda(1/p - 1/n) + (1-\lambda)\,1/r$$

Choose q=4 and p=r=2 so that $\lambda = n/4$. Then

$$\|\vec{u}(t)\|_4 \leq C\|\nabla\vec{u}(t)\|_2^{n/4} \, \|\vec{u}(t)\|_2^{1-n/4}$$

If $\vec{u}(t) \in L^2[0,T : V]$ solves the weak NS equation, then by (a)

$$\|\vec{u}(\tau)\|_2 = \|\vec{u}(\tau)\|_H \leq (\|\vec{u}_0\|_H^2 + 1/\nu\|f\|_{L^2[0T:V']}^2)^{1/2} = C_1$$

and $\quad \|\vec{u}(t)\|_4 \le C\|\nabla\vec{u}(t)\|_2^{n/4}\, C_1^{1-n/4} = C_2\|\vec{u}(t)\|_V^{n/4}$

In addition, $\quad |b(\vec{u}(t),\vec{u}(t),\vec{v})| \le \|\vec{u}\|^2_{L^4(U)}\ \|\vec{v}\|_V \quad$ implies

$$\|B[\vec{u}(t)]\|_{V'} \le \|\vec{u}\|^2_{L^4(U)} \qquad 0 \le t \le T,$$

and it follows that

$$\int_0^T \|B[\vec{u}(t)]\|^p_{V'}\, dt \le \int_0^T \|\vec{u}\|^{2p}_{L^4(U)}\, dt \le C_2 \int_0^T \|\vec{u}(t)\|_V^{np/2}\, dt$$

This last integral is finite for $np/2 \le 2$ and $\vec{u}(t) \in L^2[0,T : V]$; then we have proved that

$$\vec{u}(t) \in L^{2p}[0,T : L^4(U)^n] \quad and \qquad B[\vec{u}(t)], \vec{u}'(t) \in L^p[0,T : V'] \qquad p \le 4/n.$$

Theorem (Existence of a weak solution, $n \le 4$) For every $\vec{f} \in L^2[0,T : V']$ and each $u_0 \in H$ there exists at least one $\vec{u} \in L^2[0,T : V]$ that is a weak solution of the N-S system. In addition, this weak solution satisfies:

(a) $\vec{u} \in L^\infty[0,T : H]$ *and* $\vec{u}' \in L^{4/n}[0,T : V']$

(b) $\vec{u}(t)$ is weakly continuous in H;

i.e., $t_n \to t \in (0,T)$ implies $(\vec{u}(t_n), v)_H \to (\vec{u}(t), v)_H \quad \forall v \in H$

Proof- Let $\{\vec{w}_j\}$ denote the eigenfunctions of the Stokes operator A. Then

$$[[\vec{w}_j, \vec{w}_k]] = (A\vec{w}_j, \vec{w}_k)_H = \lambda_j\ (\vec{w}_j, \vec{w}_k)_H = 0 \quad if \ \ j \ne k$$

Then the $\vec{w}_k's$ are orthogonal in both V and in H. Let $\|\vec{w}_j\|_H = 1$ for all j.

sequence of approximate solutions
For $N = 1,2,\ldots$ let

$$\vec{u}_N(t) = \sum_{j=1}^N c_j(t;N)\, \vec{w}j$$

satisfy the approximate N-S equation,

$$(\vec{u}_N'(t),\vec{w}_k)_H + \nu[[\vec{u}_N(t),\vec{w}_k]] + b(\vec{u}_N(t),\vec{u}_N(t),\vec{w}_k) = (\vec{f},\vec{w}_k)_H$$

for $1 \le k \le N$. Then the orthogonality of the $\vec{w}_k's$ implies that

$$c_k'(t,N) + \nu\lambda_k c_k(t,N) + \sum_{i,j=1}^{N} B_{ijk}\, c_i(t;N)\, c_j(t;N) = \vec{f}_k(t)$$
$$c_k(0;N) = (\vec{u}_0,\vec{w}_k)_H$$

where

$$B_{ijk} = \int_U \vec{w}_i \bullet \nabla\vec{w}_j \bullet \vec{w}_k\, dx, \qquad \vec{f}_k(t) = (\vec{f},\vec{w}_k)_H.$$

This is a system of N nonlinear ordinary differential equations for the unknown coefficients, $c_j(t;N)$. This initial value problem has a solution on $[0,T_N)$ for each N, and, since we are going to prove a-priori bounds on $\vec{u}_N(t)$ that are uniform in N on $(0,T)$, it follows that $T_N = T$ for every N.

a-priori estimates

For each N, $\vec{u}_N(t)$ solves the approximate N-S equation, hence it follows by the same arguments used to prove lemma 2, that for every N,

(a) $||\vec{u}_N(\bullet)||^2_{C[0,T:H]} \le ||\vec{u}_0||^2_H + 1/\nu||f||^2_{L^2[0T:V']}$

(b) $||\vec{u}_N(\bullet)||^2_{L^2[0,T:V]} \le 1/\nu\,||\vec{u}_0||^2_H + 1/\nu^2\, ||f||^2_{L^2[0T:V']}$

$i.e.$, $\{\vec{u}_N(\bullet)\}$ is a bounded infinite set in $L^\infty[0,T:H]$ $and\ in$ $L^2[0,T:V]$. In addition the lemma 2 arguments imply that for $p \le 4/n$,

(c) $\{\vec{u}_N(\bullet)\}$ is a bounded infinite set in $L^{2p}[0,T:L^4(U)^n]$

(d) $\{\vec{u}_N'(\bullet)\}$ is a bounded infinite set in $L^p[0,T:V']$

It follows that there exists a subsequence (which we also denote by $\{\vec{u}_N(\bullet)\}$) such that

i) (b) implies that $\vec{u}_N(\bullet) \to \vec{u}(t) \in L^2[0,T:V]$ weakly in $L^2[0,T:V]$

ii) (a) implies that $\vec{u}_N(\bullet) \to \vec{u}(t) \in L^\infty[0,T:H]$ weak-* in $L^\infty[0,T:H]$

iii) (d) implies that $\vec{u}_N'(\bullet) \to \vec{u}'(t) \in L^p[0,T:V']$ weakly in $L^p[0,T:V']$

This last assertion relies on the fact that $p = 4/n > 1$ for n=2,3, so $L^p[0,T:V']$ is reflexive with dual space $L^q[0,T:V']$. Finally, it follows from i) and iii) and a previous lemma, that there exists a subsequence (which we also denote by $\{\vec{u}_N(\bullet)\}$) such that

iv) $\vec{u}_N(\bullet) \to \vec{u}(t) \in L^2[0,T : V]$ strongly in $L^2[0,T : H] \simeq L^2[U_T]$

passing to the limit

Now for $\vec{v}(t) \in C[0,T : V] \subset L^2[0,T : V]$ we have

$$\int_0^T(\vec{f}(t),\vec{v}(t))_H\,dt = \int_0^T\left\langle\vec{f}(t),\vec{v}(t)\right\rangle_{V'\times V}dt \qquad \text{is well defined,}$$

$$\int_0^T[[\vec{u}_N(t),\vec{v}(t)]]\,dt \to \int_0^T[[\vec{u}(t),\vec{v}(t)]]\,dt \qquad \text{(follows from i)}$$

and, since $C[0,T : V] \subset L^q[0,T : V]$

$$\int_0^T(\vec{u}_N'(t),\vec{v}(t))_H\,dt = \int_0^T\langle\vec{u}_N'(t),\vec{v}(t)\rangle_{V'\times V}\,dt \to \int_0^T(\vec{u}'(t),\vec{v}(t))_H\,dt \qquad \text{(follows from iii)}$$

Then, for $\quad C[0,T : V_N] = \left\{\sum_{i=1}^N a_i(t)\vec{w}_i,\ a_i \in C[0,T]\right\}$

it follows from the approximate N-S equation that for every $\vec{v}(t) \in C[0,T : V_N]$

$$\int_0^T\left[(\vec{u}_N'(t),\vec{v}(t))_H + \nu[[\vec{u}_N(t),\vec{v}(t)]] + b(\vec{u}_N(t),\vec{u}_N(t),\vec{v}(t)) - (\vec{f}(t),\vec{v}(t))_H\right]dt = 0$$

and it only remains to show that we can pass to the limit in the nonlinear term in order to show that $\vec{u}(t)$ is a solution of the weak N-S equation.

Note that $\quad \int_0^T(B[\vec{u}_N(t)],\vec{v}(t))_H dt = \int_0^T b(\vec{u}_N(t),\vec{u}_N(t),\vec{v}(t))dt$

$$= -\int_0^T b(\vec{u}_N(t),\vec{v}(t),\vec{u}_N(t))dt$$

$$= -\int_0^T\sum_{i,j=1}^n\int_U \vec{u}_{i,N}(t)\,\partial_i\vec{v}_j(t)\,\vec{u}_{j,N}(t)\,dx\ dt$$

where $\vec{u}_{j,N}(t)$ denotes the j-th component of $\vec{u}_N(t)$. Now it follows from iv) that $\vec{u}_{j,N}(t) \to \vec{u}_j(t)$ strongly in $L^2(U_T)$ which implies, in turn, that

$$\vec{u}_{j,N}^2(t) \to \vec{u}_j^2(t) \quad \text{and} \quad \vec{u}_{j,N}(t)\,\vec{u}_{i,N}(t) \to \vec{u}_j(t)\,\vec{u}_i(t) \ \text{ as } N \to \infty$$

where the convergence is strong convergence in $L^1(U_T)$. i.e.,

$$\|\vec{u}_{j,N}^2(t) - \vec{u}_j^2(t)\|_{L^1(U_T)} = \int_0^T\int_U|\vec{u}_{j,N}^2(t) - \vec{u}_j^2(t)|\,dx\,dt$$

$$= \int_0^T \int_U |\vec{u}_{j,N}(t) - \vec{u}_j(t)||\vec{u}_{j,N}(t) + \vec{u}_j(t)|\,dx\,dt$$

$$\leq \|\vec{u}_{j,N} + \vec{u}_j\|_{L^2(U_T)} \|\vec{u}_{j,N} - \vec{u}_j\|_{L^2(U_T)} \leq C\|\vec{u}_{j,N} - \vec{u}_j\|_{L^2(U_T)}$$

and,

$$\|\vec{u}_{j,N}(t)\,\vec{u}_{i,N}(t) \to \vec{u}_j(t)\,\vec{u}_i(t)\|_{L^1(U_T)} \leq$$

$$\leq \|\vec{u}_{i,N}\|_{L^2(U_T)} \|\vec{u}_{j,N} - \vec{u}_j\|_{L^2(U_T)} + \|\vec{u}_j\|_{L^2(U_T)} \|\vec{u}_{i,N} - \vec{u}_i\|_{L^2(U_T)}$$

Now we combine these results to conclude that as $N \to \infty$,

$$\left| \int_0^T (B[\vec{u}_N(t)] - B[\vec{u}(t)], \vec{v}(t))_H dt \right| =$$

$$= \left| \int_0^T \sum_{i,j=1}^n \int_U [\vec{u}_i(t)\,\vec{u}_{j,N}(t) - \vec{u}_{i,N}(t)\vec{u}_{j,N}(t)]\partial_i \vec{v}_j(t)\,dx\;dt \right|$$

$$\leq \sum_{i,j=1}^n \|\vec{u}_i(t)\,\vec{u}_{j,N}(t) - \vec{u}_{i,N}(t)\vec{u}_{j,N}(t)\|_{L^1(U_T)} \|\partial_i \vec{v}_j(t)\|_\infty \to 0$$

It follows now that

$$\int_0^T \Big[(\vec{u}'(t), \vec{v}(t))_H + \nu((\vec{u}(t), \vec{v}(t))) + b(\vec{u}(t), \vec{u}(t), \vec{v}(t)) - (\vec{f}(t), \vec{v}(t))_H \Big] dt = 0$$

This holds for every $\vec{v} \in C[0,T : V_N] = \left\{ \sum_{i=1}^N a_i(t)\vec{w}_i,\ a_i \in C[0,T] \right\}$ for every N, hence it must hold for every $\vec{v} \in C[0,T : V]$, and by continuity, for every $\vec{v} \in L^2[0,T : V]$.
This completes the proof that the weak limit of the sequence of approximate solutions is a solution of the weak N-S equation. Using a test function $\vec{v} \in C^1[0,T : V]$, with $v(T) = 0$ and integrating the time derivative term by parts leads to the result that the weak solution must satisfy the initial condition, $\vec{u}(0) = \vec{u}_o$ by the same argument that we used for the linear parabolic problems. This completes the proof that the N-S equation has at least one weak solution.

The weak continuity of $\vec{u}(t)$ is a result of the facts:

(i) $\vec{u}_N(\bullet) \to \vec{u}(t)$ weak-* in $L^\infty[0,T : H]$ $\;so\; \vec{u}(t) \in L^\infty[0,T : H]$

(ii) $\vec{u}(t) \in L^2[0,T : V] \quad and \quad \vec{u}'(t) \in L^{4/n}[0,T : V'] \quad so \quad \vec{u}(t) \in C[0,T : V']$

(iii) **Lemma**- If X and Y are two Banach spaces with X continuously embedded in Y and if $u \in L^\infty[0,T : X]$ is weakly continuous t with values in Y, then u is weakly continuous t with values in X.

In the case n=2 we can show that the weak solution is unique and belongs to $C[0,T : H]$.
In the case n=3 the weak solution has not been shown to be unique and the weak solution is only weakly continuous from $[0,T]$ to H.