The Stationary Navier-Stokes Equations

Having considered the linear Stokes equations, we will now bring back the nonlinear term and consider the nonlinear version of the Stokes system. The solution of these equations can be viewed as the limit to which the solution of the full N-S equations tends as t tends to infinity. Of course no one knows at this point if the solutions of the N-S equations do tend to some limit as t tends to infinity.

Let $U \subset R^n$ $(n = 2,3)$ be open and bounded with a smooth boundary ∂U. Let $\vec{f} \in L^2(U)^n$ be given and consider

$$\begin{aligned} -\nu\nabla^2\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p &= \vec{f} \qquad in\ U \subset R^n \\ div\ \vec{u} &= 0, \qquad in\ U, \\ \vec{u} &= 0 \qquad on\ \partial U. \end{aligned} \qquad (1)$$

If we define $$b(\vec{u},\vec{v}.\vec{w}) = ((\vec{u}\cdot\nabla\vec{v},\vec{w}))_0 = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_U u_i\,\partial_i v_j\,w_j\,dx,$$

then we will say that $\vec{u} \in V$ is a weak solution of (1) if

$$\nu[[\vec{u},\vec{v}]] + b(\vec{u},\vec{u},\vec{v}) = ((\vec{f},\vec{v}))_0 \qquad \forall\vec{v} \in V \qquad (2)$$

Before we can prove the existence of a weak solution for (2), we will have to establish a number of properties of the trilinear form b. We recall the following inequalities:

(a) Holder inequality-

if $u \in L^p(U),\ v \in L^q(U),\ \ \frac{1}{p} + \frac{1}{q} = 1,\ then\ \ uv \in L^1(U)\ and$

$$\int_U |uv|\,dx \le \|u\|_p\,\|v\|_q$$

(b) Extended Holder inequality-

if $u \in L^p(U),\ v \in L^q(U),\ w \in L^r(U),\ \ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,\ then\ \ uvw \in L^1(U)\ and$

$$\int_U |uvw|\,dx \le \|u\|_p\,\|v\|_q\,\|w\|_r$$

We recall, also that

$$H_\alpha(U) \subset W_0^{m,p}(U) \qquad if \qquad 2\alpha > m,\quad and \quad 2\alpha - \frac{n}{2} > m - \frac{n}{p}$$

$$H_\alpha(U) \subset C(\bar{U}) \qquad if \qquad 2\alpha > \frac{n}{2}.$$

It is also true that $W_0^{1,p}(U) \subset L^q(U)$ is a compact embedding if:

$$1 \le p < n \quad and\ 1 \le q \le \frac{n-p}{np} = \frac{1}{p} - \frac{1}{n}\ ,$$

or

$$p \ge n \qquad and \qquad 1 \le q < \infty.$$

In particular, we have

for $n = 2 \qquad \|u\|_{L^q(U)} \le C\|u\|_{H_0^1(U)} \qquad 1 \le q < \infty,$

for $n = 3$ $\quad \|u\|_{L^q(U)} \leq C\|u\|_{H_0^1(U)} \quad 1 \leq q \leq 6,$

for $n \geq 3$ $\quad \|u\|_{L^q(U)} \leq C\|u\|_{H_0^1(U)} \quad 1 \leq q \leq \dfrac{2n}{n-2}.$

Finally, recall that $\quad \|u\|_{H_0^1(U)} = \|\nabla u\|_0 = \|u\|_{1/2}.$

Lemma 1- The trilinear form $b(\vec{u},\vec{v}.\vec{w})$ is continuous on $V \times V \times V$ for $n \leq 4$.

Proof- Suppose $\vec{u},\vec{v}.\vec{w} \in V$. Then for $n = 3,4$

$$u_i \in L^{2n/(n-2)}(U), \quad \partial_i v_j \in L^2(U), \quad w_j \in L^n(U),$$

for $1 \leq i,j \leq n$. Then the extended Holder inequality implies

$$\left|\int_U u_i \partial_i v_j w_j \, dx\right| \leq \|u_i\|_p \|\partial_i v_j\|_2 \|w_j\|_n$$

since $\quad \dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = \dfrac{n-2}{2n} + \dfrac{1}{2} + \dfrac{1}{n} = 1.$

Then the embedding $H_0^1(U) \subset L^q(U)$ $1 \leq q \leq \dfrac{2n}{n-2}$, implies

$$|b(\vec{u},\vec{v},\vec{w})| \leq C\|\vec{u}\|_V \|\vec{v}\|_V \|\vec{w}\|_V.$$

so $b(\vec{u},\vec{v}.\vec{w})$ is continuous on $V \times V \times V$ for $n = 3,4$.

For $n = 2$ $H_0^1(U) \subset L^q(U)$ for $1 \leq q < \infty$ so we choose $p = r = 4,\ q = 2$ so that

$$\left|\int_U u_i \partial_i v_j w_j \, dx\right| \leq \|u_i\|_4 \|\partial_i v_j\|_2 \|w_j\|_4 \qquad \left(\tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{4} = 1\right)$$

and $b(\vec{u},\vec{v}.\vec{w})$ is continuous on $V \times V \times V$ for $n = 2$ as well■.

Note that $H_0^1(U) \subset L^4(U)$ for $n \leq 4$ so $\forall \vec{u},\vec{v},\vec{w} \in V$

$$|b(\vec{u},\vec{v},\vec{w})| \leq C\|\vec{u}\|_{L^4(U)} \|\nabla\vec{v}\|_{L^2(U)} \|\vec{w}\|_{L^4(U)} \leq C\|\vec{u}\|_V \|\vec{v}\|_V \|\vec{w}\|_V.$$

and

$$|b(\vec{u},\vec{u},\vec{w})| \leq C\|\vec{u}\|_V^2 \, \|\vec{w}\|_V..$$

Then for $\vec{u} \in V$, fixed, $F(\vec{v}) = b(\vec{u},\vec{u},\vec{v})$ is a bounded linear functional on V and $\forall \vec{v} \in V$,

$$F(\vec{v}) = \langle B(\vec{u}),\vec{v}\rangle_{V' \times V} \qquad \text{for some } B(\vec{u}) \in V'$$

$$= (B(\vec{u}),\vec{v})_H \qquad \text{because H is the pivot space for } V,\ V'.$$

Then the weak problem (2) is equivalent to

$$A\vec{u} + B(\vec{u}) - \vec{f} = 0 \qquad in \quad V' \qquad\qquad (3)$$

where $A = -\nu P \nabla^2$ denotes the Stokes operator. If $\vec{u} \in V$ solves (3) then, since (3) implies that $A\vec{u} + B(\vec{u}) - \vec{f} \in K^0 \cap H^{-1}(U)^n$, there exists a $p \in L^2(U)$ such that

$$\nabla p = A\vec{u} + B(\vec{u}) - \vec{f}.$$

This clarifies the sense in which (1) is satisfied by the weak solution for (2).

Lemma 2- For $U \subset R^n$ open and bounded

(a) $b(\vec{u},\vec{v},\vec{v}) = 0 \qquad \forall \vec{u} \in V, \quad \vec{v} \in H_0^1(U)^n$

(b) $b(\vec{u},\vec{v},\vec{w}) = -b(\vec{u},\vec{w},\vec{v}) \qquad \forall \vec{u} \in V, \quad \vec{v},\vec{w} \in H_0^1(U)^n$.

Proof- It is sufficient to show this for $\vec{u} \in K$ *and* $v \in C_0^\infty(U)^n$. Then

$$\int_U u_i \partial_i v_j v_j \, dx = \tfrac{1}{2} \int_U u_i \partial_i (v_j v_j) \, dx = -\tfrac{1}{2} \int_U (v_j)^2 \partial_i u_i \, dx,$$

and $\quad b(\vec{u},\vec{v},\vec{v}) = -\frac{1}{2} \sum_{j=1}^{n} \int_U (v_j)^2 \, div \vec{u} \, dx = 0.$

i.e., $\quad b(\vec{u},\vec{v},\vec{v}) = \int_U \vec{u} \cdot \nabla \vec{v} \cdot \vec{v} \, dx = \int_U \vec{u} \cdot \nabla (\frac{1}{2} \vec{v} \cdot \vec{v}) \, dx = -\frac{1}{2} \int_U div \vec{u} \, (\vec{v} \cdot \vec{v}) \, dx = 0.$

Now (a) implies

$$0 = b(\vec{u},\vec{v}+\vec{w},\vec{v}+\vec{w}) = b(\vec{u},\vec{v},\vec{v}) + b(\vec{u},\vec{v},\vec{w}) + b(\vec{u},\vec{w},\vec{v}) + b(\vec{u},\vec{w},\vec{w})$$

and $\qquad 0 = b(\vec{u},\vec{v},\vec{w}) + b(\vec{u},\vec{w},\vec{v}) \blacksquare$

Fixed Point Theorems

The existence of weak solutions for (1) can be proved using various fixed point theorems. We will review some fixed point theorems now.

A 1-dimensional fixed point theorem- Suppose $f \in C[0,1]$ *and* $f[0,1] \subset [0,1]$. Then there exists an $x_0 \in [0,1]$ *such that* $f(x_0) = x_0$.

Proof- Let $g(x) = f(x) - x$. Then $g \in C[0,1]$ *and* $g(0) = f(0) \geq 0$, $g(1) = f(1) - 1 \leq 0$. If $g(0) = 0$ *then* $x_0 = 0$ *and if* $g(1) = 0$ *then* $x_0 = 1$. If neither $g(0) = 0$ *nor* $g(1) = 0$ *then* $g(0) > 0$ *and* $g(1) < 0$ and since g is continuous on $[0,1]$, there exists at least one $x_0 \in [0,1]$ such that $g(x_0) = 0 = f(x_0) - x_0$. $\blacksquare$

Note that the only hypotheses here were that *f* is continuous on $I = [0,1]$ *and* $f[I] \subset I$. More generally, we have

Brouwer's Fixed Point Theorem- Suppose B is a closed ball in R^n and $f : B \to B$ is continuous. Then there exists an $x_0 \in B$ *such that* $f(x_0) = x_0$.

Recall the following definitions:

- A set A in Banach space X is *compact* if every sequence $\{x_n\} \subset A$ contains a subsequence converging in the norm of X to an element of A.
- A linear mapping $T : X \to X$ is *compact* if, for every bounded sequence $\{x_n\} \subset X$, it follows that $\{Tx_n\}$ contains a convergent subsequence; i.e., $\|x_n\|_X \leq M$ *implies* $\exists \{x_m\} \subset \{x_n\}$ *such that* $Tx_m \to x \in X$.
- A set A in X is *convex* if, for all $x,y \in A$, $tx + (1-t)y \in A$ *for all* $t \in [0,1]$.

In both the previous theorems the mapping f is continuous and the sets I and B are compact and convex. The Brouwer theorem generalizes to

Schauder's Fixed Point Theorem- Let A denote a compact convex set in a Banach space X. Suppose $T : A \to A$ is continuous. Then there exists an $x_0 \in A$ *such that* $T(x_0) = x_0$.

The idea of the proof is to approximate T by maps $T_n : R^n \to R^n$ and use the Brouwer theorem.

Corollary- Suppose A is closed and convex, $T : A \to A$ is continuous, and $T[A]$ has compact closure. Then there exists an $x_0 \in A$ *such that* $T(x_0) = x_0$.

We can avoid any reference to a specific set A by assuming compactness for the mapping, T, together with the conidition that the fixed points of the family of mappings, $\{tT,\ 0 \le t \le 1\}$, all lie in a bounded set. The result is the

Leray-Schauder Fixed Point Theorem- Suppose the mapping $T : X \to X$ is compact and, for any $t \in [0,1]$, $x = tTX$ implies $\|x\|_X \le M$. Then there exists an $x_0 \in X$ *such that* $T(x_0) = x_0$.

The Leray-Schauder theorem can be proved from the corollary to the Schauder theorem. It can be used to prove existence theorems for linear elliptic and parabolic partial differential equations having variable coefficients. In such proofs the Banach spaces are the so called Schauder spaces, $C^{2+\alpha}(U)$. We will show how existence for weak solutions to the Stokes system can be proved using either the Brouwer theorem or the Leray-Schauder theorem.

An Existence Proof Using the Brouwer Theorem

Given an $f \in H$ *or* $f \in H^{-1}(U)^n$, we will show there exists at least one $\vec{u} \in V$ such that

$$\nu[[\vec{u},\vec{v}]] + b(\vec{u},\vec{u},\vec{v}) = ((\vec{f},\vec{v}))_0 \qquad \forall \vec{v} \in V$$

or, equivalently, $\nu A\vec{u} + B(\vec{u}) - \vec{f} = 0 \quad in \quad V'$.

Then there exists a $p \in L^2(U)$ satisfying, $\nabla p = \nu A\vec{u} + B(\vec{u}) - \vec{f}$.

Proof- Let $\{\vec{w}_j\}$ denote a countable orthonormal basis for V (e.g., the eigenfunctions of the Stokes operator). Then for each integer N, let

$$\vec{u}_N = \sum_{i=1}^{N} C_{i,N}\vec{w}_i$$

where the coefficients $\{C_{i,N}\}$ are chosen so that

$$\nu[[\vec{u}_N,\vec{v}]] + b(\vec{u}_N,\vec{u}_N,\vec{v}) = ((\vec{f},\vec{v}))_0 \quad for\ all\ \vec{v} \in V_N = span\{\vec{w}_1,\ldots,\vec{w}_N\}.$$

This is a nonlinear system of algebraic equations for the N unknowns $\{C_{i,N}\}$. Existence of a solution for this system will follows from

Lemma A- Let X denote a finite dimensional Hilbert space with inner product and associated norm denoted by $(c,d)_X$ *and* $|c|_X$, respectively. Let P denote a mapping of X into itself such that P is continuous and satisfies

$$(P(c),c)_X > 0 \text{ for all } c \text{ such that } |c|_X = k, \quad k > 0$$

Then there exists a $c \in X$ *such that* $|c|_X \le k$, *and* $P(c) = 0$.

i) *Existence of an approximate solution*- Let us assume for the moment that Lemma A holds, and let $X = V_N$ *equipped with* $(\vec{u},\vec{v})_X = [[\vec{u},\vec{v}]]$. Further, let $P : X \to X$ be defined by

$$(P(\vec{u}),\vec{v})_X = [[P(\vec{u}),\vec{v}]] = \nu[[\vec{u},\vec{v}]] + b(\vec{u},\vec{u},\vec{v}) - ((\vec{f},\vec{v}))_0.$$

Since $\quad F(\vec{v}) := b(\vec{u},\vec{u},\vec{v}) - ((\vec{f},\vec{v}))_0.$

satisfies $\quad |F(\vec{v})| \le \left(C\|\vec{u}\|_V^2 + \|f\|_H\right)\|\vec{v}\|_V,$

it follows from the Riesz theorem (not availing ourselves of the pivot space H), that

$$F(\vec{v}) = [[z_F(\vec{u}),\vec{v}]] \qquad \forall\vec{v} \in V.$$

Then $\quad (P(\vec{u}),\vec{v})_X = [[\nu\vec{u} + z_F(\vec{u}),\vec{v}]] = (\nu\vec{u} + z_F(\vec{u}),\vec{v})_X \qquad \forall\vec{u} \in X$

so $P : X \to X$ is continuous. Moreover,

$$\begin{aligned}(P(\vec{u}),\vec{u})_X &= \nu[[\vec{u},\vec{u}]] + b(\vec{u},\vec{u},\vec{u}) - ((\vec{f},\vec{u}))_0.\\ &\ge \nu\|\vec{u}\|_V^2 - \|\vec{f}\|_{V'}\|\vec{u}\|_V \ge \|\vec{u}\|_V\left(\nu\|\vec{u}\|_V - \|\vec{f}\|_{V'}\right)\end{aligned}$$

and so, for $\|\vec{u}\|_V = k$ *and* $k > \dfrac{\|\vec{f}\|_{V'}}{\nu}$, we have $(P(\vec{c}),\vec{c})_X > 0$. Then the hypotheses of lemma A are satisfied and it follows that for each integer, N, there exists at least one $\vec{u}_N$ *such that* $P(\vec{u}_N) = 0$; i.e., $\vec{u}_N$ is an approximate solution for the nonlinear Stokes system. We must now show that the sequence of approximate solutions converges to some limit $\vec{u}$ *in* V and that this limit, $\vec{u}$, is then a weak solution for the Stokes system.

ii) *An A-priori Estimate*- For each integer, N, we have

$$\nu[[\vec{u}_N,\vec{u}_N]] + b(\vec{u}_N,\vec{u}_N,\vec{u}_N) = ((\vec{f},\vec{u}_N))_0$$

and $\quad \nu[[\vec{u}_N,\vec{u}_N]] = ((\vec{f},\vec{u}_N))_0 \le \|\vec{f}\|_{V'}\|\vec{u}_N\|_V$

since $\quad b(\vec{u}_N,\vec{u}_N,\vec{u}_N) = 0.$

Then $\quad \|\vec{u}_N\|_V \le \dfrac{1}{\nu}\|\vec{f}\|_{V'} \qquad \forall N.$

iii) *Passing to the limit*- Since the sequence $\{\vec{u}_N\}$ is bounded in V, uniformly (in N), it follows that there exists a subsequence $\{\vec{u}_{N'}\} \subset \{\vec{u}_N\}$such that $\{\vec{u}_{N'}\}$ converges weakly to some $\vec{u} \in V$. Since V is compactly embedded in $H = L^2(U)^n$, there exists a subsequence of the subsequence that converges strongly, in the norm of H, to the same limit. That is, $\|\vec{u}_{N''} - \vec{u}\|_H \to 0$. Now suppose we can prove

Lemma B- If $\{\vec{u}_{N'}\}$ converges weakly in V and strongly in H to some $\vec{u} \in V$, then necessarily,

$$b(\vec{u}_N,\vec{u}_N,\vec{v}) \to b(\vec{u},\vec{u},\vec{v}) \quad \forall \vec{v} \in K.$$

Assuming, for the moment, that this lemma holds, then

$$\begin{array}{lllllll} v[[\vec{u}_N,\vec{v}]] & + & b(\vec{u}_N,\vec{u}_N,\vec{v}) & = & ((\vec{f},\vec{v}))_0 & \text{for all } \vec{v} \in V_N = span\{\vec{w}_1,\ldots,\vec{w}_N\}. \\ \downarrow & & \downarrow & & \downarrow & \qquad\qquad \downarrow \quad \text{as } N \to \infty \\ v[[\vec{u},\vec{v}]] & + & b(\vec{u},\vec{u},\vec{v}) & = & ((\vec{f},\vec{v}))_0 & \text{for all } \vec{v} \in \bigcup_{N\geq 0} V_N = V \end{array}$$

This proves that the sequence of approximate solutions $\{\vec{u}_N\}$ contains a subsequence which converges to some $\vec{u} \in V$ (weakly in V but strongly in H) and this limit point is a weak solution of the Stokes system. Note that weak convergence of the sequence of approximate solutions was sufficient for passing to the limit in the linear problems of last semester, whereas here, in the nonlinear term, $b(u,u,v)$, strong convergence is required.

It remains only to prove lemmas A and B.

Proof of lemma A-

The lemma follows from the Brouwer fixed point theorem:

Suppose $f : B_k(0) \to B_k(0) = \{\vec{x} \in R^n : \|\vec{x}\| \leq k\}$ *is continuous.*

Then f *has at least one fixed point in* $B_k(0)$.

Suppose $P(c)$ has no zero in $B_k(0) = \{c \in X : |c|_X \leq k\}$. Then if we define

$$c \to S(c) := -k\frac{P(c)}{|P(c)|_X}$$

it is evident that S maps $B_k(0)$ *into* $B_k(0)$ continuously. Then by the Brouwer theorem $S(c)$ has a fixed point in $B_k(0)$; i.e., there exists $c_0 \in B_k(0)$ such that

$$c_0 = -k\frac{P(c_0)}{|P(c_0)|_X}.$$

Then

$$|c_0|_X = \left| -k\frac{P(c_0)}{|P(c_0)|_X} \right|_X = k.$$

But this leads to $\quad (c_0,c_0)_X = |c_0|_X^2 = k^2 = -k\dfrac{(P(c_0),c_0)_X}{|P(c_0)|_X}$

i.e., $\quad (P(c_0),c_0)_X = -k\,|P(c_0)|_X < 0$

which contradicts the assumption on P. The contradiction shows $P(c)$ has a zero in $B_k(0)$.■

Proof of lemma B-

From a previous lemma, we have

$$b(\vec{u}_N,\vec{u}_N,\vec{v}) = -b(\vec{u}_N,\vec{v},\vec{u}_N) = -\sum_i\sum_j\int_U(\vec{u}_N)_i\partial_i v_j\,(\vec{u}_N)_j\,dx$$

Then for each N,

$$\int_U |(\vec{u}_N)_i \partial_i v_j (\vec{u}_N)_j - u_i\ \partial_i v_j u_j|dx \le \int_U |[(\vec{u}_N)_i - u_i]\partial_i v_j (\vec{u}_N)_j|dx\ \int_U |u_i\ \partial_i v_j[(\vec{u}_N)_j - u_j]|dx$$

But $\partial_i v_j \in L^\infty(U)$ if $\vec{v} \in K$ *and* $\|\vec{u}_N - \vec{u}\|_{L^2(U)^n} \to 0,$ hence

$$\int_U |(\vec{u}_N)_i \partial_i v_j (\vec{u}_N)_j - u_i\ \partial_i v_j u_j|dx \le \|\vec{u}_N - \vec{u}\|_{L^2(U)^n} \|\partial_i v_j\|_\infty \frac{2\|f\|_{V'}}{\nu} \to 0 \ as\ N \to \infty.$$

Then $\quad b(\vec{u}_N, \vec{v}, \vec{u}_N) \to b(\vec{u}, \vec{v}, \vec{u}) = -b(\vec{u}, \vec{u}, \vec{v}) \qquad \forall \vec{v} \in K$

and the result extends to V by continuity.■

An Existence Proof Using the Leray-Schauder Theorem

If we fix $\vec{w} \in V$ and consider the linear problem

$$\nu[[\vec{u}, \vec{v}]] = ((\vec{f}, \vec{v}))_0 - b(\vec{w}, \vec{w}, v) \qquad \forall \vec{v} \in V,$$

Then $\qquad F(\vec{v}) := ((\vec{f}, \vec{v}))_0 - b(\vec{w}, \vec{w}, v) \qquad \forall \vec{v} \in V,$

defines a bounded linear functional on V; i.e.,

$$\begin{aligned} |F(\vec{v})| &\le \|\vec{f}\|_{V'} \|\vec{v}\|_V + |b(\vec{w}, \vec{w}, v)| \\ &\le (\|\vec{f}\|_{V'} + C\|\vec{w}\|_V^2)\|\vec{v}\|_V \end{aligned}$$

It follows that there exists a unique $\vec{z}(\vec{w}) \in V$ such that

$$\nu[[\vec{u}, \vec{v}]] = [[\vec{z}(\vec{w}), \vec{v}]] \qquad \forall \vec{v} \in V,$$

so

$$\vec{u} = \frac{1}{\nu}\vec{z}(\vec{w}).$$

If we define $\quad T\vec{w} = \frac{1}{\nu}\ \vec{z}(\vec{w})\ $ from V into V, then

i) $T : V \to V \quad$ *is compact*

ii) If $\vec{u} \in V$ *is such that* $\vec{u} = tT\vec{u}$ for $t \in [0, 1],$
then $\|\vec{u}\|_V \le M$ for some $M > 0$ not depending on $\vec{u}.$

Assuming for the moment that i) and ii) both hold, then the Leray-Schauder theorem implies that T has a fixed point $\vec{u}$ *in* $V.$ Then the fixed point satisfies $T\vec{u} = \frac{1}{\nu}\ \vec{z}(\vec{u}),\ i.e.,$

$$\begin{aligned} [[\vec{u}, \vec{v}]] &= \left[\left[\ \frac{1}{\nu}\vec{z}(\vec{u}), \vec{v}\right]\right] \qquad \forall \vec{v} \in V, \\ &= F(\vec{v}) = ((\vec{f}, \vec{v}))_0 - b(\vec{u}, \vec{u}, v) \qquad \forall \vec{v} \in V, \end{aligned}$$

which is to say, $\vec{u}$ is a weak solution for the nonlinear Stokes system. Note that T is not a contraction so the fixed point need not be unique. However, for $\vec{f}$ sufficiently small, (or for ν sufficiently large) the inverse function theorem could be applied to show that there is a unique solution in a small ball about the origin. Alternatively, we can (and will) prove uniqueness by arguments like those employed for linear elliptic equations. It remains now only to prove i) and ii).

Proof of i)

Suppose $\{\vec{w}_m\} \subset V$ is bounded; $i.e.,\ \|\vec{w}_m\|_V \leq M\ \forall m.$ Then

$$\nu[[T\vec{w}_m - T\vec{w}_n, v]] = b(\vec{w}_n, \vec{w}_n, \vec{v}) - b(\vec{w}_m, \vec{w}_m, \vec{v})$$

and

$$\nu|\,[[T\vec{w}_m - T\vec{w}_n, v]]| \leq C\|\vec{w}_n - \vec{w}_m\|^2_{L^4(U)^n}\|\vec{v}\|_V.$$

The embedding $V \subset L^4(U)^n$ is compact for $n \leq 4$ hence $\{\vec{w}_m\}$ contains a subsequence $\{\vec{w}_{m'}\}$ such that $\|\vec{w}_{m'} - \vec{w}_{m''}\|^2_{L^4(U)^n} \to 0.$ Choosing $\vec{v} = T\vec{w}_{m'} - T\vec{w}_{m''}$ leads to

$$\nu\|T\vec{w}_{m'} - T\vec{w}_{m''}\|^2_V \leq C\|\vec{w}_{m'} - \vec{w}_{m''}\|^2_{L^4(U)^n}\|T\vec{w}_{m'} - T\vec{w}_{m''}\|_V$$

and this implies that $\{T\vec{w}_{m'}\}$ is strongly convergent in V. Then T maps bounded sequences into relatively compact ones, which is to say, T is compact.■

Proof of ii)

Suppose $\vec{u} = tT\vec{u}\ \mathit{for\ some}\ \vec{u} \in V$ and $t \in [0,1].$ Then

$$\nu[[\vec{u},\vec{v}]] = t[[\vec{z}(\vec{u}),\vec{v}]] = t\{((\vec{f},\vec{v}))_0 - b(\vec{u},\vec{u},v)\} \qquad \forall\vec{v} \in V.$$

Choose $\vec{v} = \vec{u}.$ Then

$$\nu\|\vec{u}\|^2_V = \left|t\Big(((\vec{f},\vec{v}))_0 - b(\vec{u},\vec{u},\vec{u})\Big)\right| \leq t\|\vec{f}\|_{V'}\|\vec{u}\|_V$$

and

$$\|\vec{u}\|_V \leq \frac{t}{\nu}\ \|\vec{f}\|_{V'} \leq \frac{1}{\nu}\ \|\vec{f}\|_{V'} = M.$$■

Uniqueness

We have succeeded in showing the existence of weak solutions to the stationary N-S equations by means of fixed point arguments. Of course such arguments say nothing about the uniqueness of the solutions. It is typical of nonlinear problems that uniqueness is a separate issue from existence, even in the case of stationary problems (for linear stationary problems recall that existence and uniqueness always occured together.)

Theorem (**Uniqueness of Weak Solutions**) If $\nu > 0$ is sufficiently large, or $\vec{f} \in V'$ is sufficiently small, then the stationary N-S equations have at most one weak solution.

Proof- For $n \leq 4$ we have

$$|b(\vec{u},\vec{v},\vec{w})| \leq C\|\vec{u}\|_V\|\vec{v}\|_V\|\vec{w}\|_V \qquad \forall\vec{u},\vec{v},\vec{w} \in V.$$

We will show that the weak solution of the stationary N-S equations is unique if

$$\nu^2 > C\|\vec{f}\|_{V'}. \qquad (*)$$

Suppose $\vec{u}$ is any weak solution of the stationary N-S equations. Then, as we saw in the proof of ii) in the previous section

$$\|\vec{u}\|_V \leq \frac{1}{\nu}\, \|\vec{f}\|_{V'}.$$

Now suppose $\vec{u}_1, \vec{u}_2$ are both weak solutions. Then

$$\nu[[\vec{u}_1, \vec{v}]] = ((\vec{f}, \vec{v}))_0 - b(\vec{u}_1, \vec{u}_1, v) \qquad \forall \vec{v} \in V,$$

$$\nu[[\vec{u}_2, \vec{v}]] = ((\vec{f}, \vec{v}))_0 - b(\vec{u}_2, \vec{u}_2, v) \qquad \forall \vec{v} \in V,$$

and

$$\begin{aligned}\nu[[\vec{u}_1 - \vec{u}_2, \vec{v}]] &= b(\vec{u}_2, \vec{u}_2, v) - b(\vec{u}_1, \vec{u}_1, v) \\ &= b(\vec{u}_2, \vec{u}_2, v) - b(\vec{u}_2, \vec{u}_1, v) + b(\vec{u}_2, \vec{u}_1, v) - b(\vec{u}_1, \vec{u}_1, v) \\ &= b(\vec{u}_2, \vec{u}_2 - \vec{u}_1, v) - b(\vec{u}_2 - \vec{u}_1, \vec{u}_1, v).\end{aligned}$$

Now choose $\vec{v} = \vec{u}_2 - \vec{u}_1,\ so\ b(\vec{u}_2, \vec{u}_2 - \vec{u}_1, v) = 0$. Then

$$\nu\|\vec{u}_2 - \vec{u}_1\|_V^2 = -b(\vec{u}_2 - \vec{u}_1, \vec{u}_1, \vec{u}_2 - \vec{u}_1) = b(\vec{u}_2 - \vec{u}_1, \vec{u}_2 - \vec{u}_1, \vec{u}_1),$$

and

$$\nu\|\vec{u}_2 - \vec{u}_1\|_V^2 \leq C\|\vec{u}_2 - \vec{u}_1\|_V^2\, \|\vec{u}_2\|_V \leq \frac{C}{\nu}\, \|\vec{f}\|_V \|\vec{u}_2 - \vec{u}_1\|_V^2$$

i.e., $$\left(\nu - \frac{C}{\nu}\, \|\vec{f}\|_V\right) \|\vec{u}_2 - \vec{u}_1\|_V^2 \leq 0.$$

If $\nu^2 > C\, \|\vec{f}\|_V$ then it follows that $\|\vec{u}_2 - \vec{u}_1\|_V^2 = 0$ which is the desired uniqueness result. Obviously if $\nu^2 \leq C\, \|\vec{f}\|_V$, or if $n > 4$ (where the estimate on $b(\vec{u}, \vec{u}, \vec{v})$ may fail) the solution need not be unique.■

A Nonuniqueness Result

Suppose $U \subset R^n$ is open and bounded with smooth boundary ∂U. Let $\vec{a} = \vec{a}(\vec{x})$ denote a given, smooth divergence free vector field in U, and for $\mu > 0\ let\ \vec{\phi}, \pi$ denote a nontrivial solution pair for

$$\begin{aligned}\frac{1}{\mu}\nabla^2\vec{\phi} &= \vec{a} \bullet \nabla\vec{\phi} + \vec{\phi} \bullet \nabla\vec{a} + \nabla\pi && in\ U \\ div\,\vec{\phi} &= 0 && in\ U, \\ \vec{\phi} &= \vec{0} && on\ \partial U.\end{aligned} \qquad (1)$$

Then we will show there exist vector fields $\vec{f}, \vec{g}$ such that

$$\begin{aligned}-\nu\nabla^2\vec{u} + \vec{u} \bullet \nabla\vec{u} + \nabla p &= \vec{f} && in\ U \\ div\,\vec{u} &= 0 && in\ U, \\ \vec{u} &= \vec{g} && on\ \partial U\end{aligned} \qquad (2)$$

admits two distinct solutions for the data $\vec{f}, \vec{g}$ with $\nu = \frac{1}{2\mu}$.

Let $$\vec{u}_1 = \tfrac{1}{2}\left(\vec{a} + \vec{\phi}\right), \qquad \vec{u}_2 = \tfrac{1}{2}\left(\vec{a} - \vec{\phi}\right).$$

Then $\vec{u}_1$ and $\vec{u}_2$ both solve (2) for

$$\vec{g} = \tfrac{1}{2}\vec{a}|_{\partial U} \qquad \textit{and} \qquad \vec{f} = -\frac{1}{4\mu}\nabla^2\vec{a} + \tfrac{1}{4}\left(\vec{a}\cdot\nabla\vec{a} + \vec{\phi}\cdot\nabla\vec{\phi}\right).$$

That is,

$$-\frac{1}{2\mu}\nabla^2\vec{u}_1 + \vec{u}_1\cdot\nabla\vec{u}_1 = -\frac{1}{4\mu}\nabla^2\left(\vec{a}+\vec{\phi}\right) + \tfrac{1}{4}\left(\vec{a}+\vec{\phi}\right)\cdot\nabla\left(\vec{a}+\vec{\phi}\right)$$
$$= -\frac{1}{4\mu}\nabla^2\vec{a} + \tfrac{1}{4}\left(\vec{a}\cdot\nabla\vec{a} + \vec{\phi}\cdot\nabla\vec{\phi}\right) - \frac{1}{4\mu}\nabla^2\vec{\phi} + \tfrac{1}{4}\left(\vec{a}\cdot\nabla\vec{\phi} + \vec{\phi}\cdot\nabla\vec{a}\right),$$

and then it follows from (1) that

$$-\frac{1}{2\mu}\nabla^2\vec{u}_1 + \vec{u}_1\cdot\nabla\vec{u}_1 = \vec{f} - \tfrac{1}{4}\nabla\pi.$$

Similarly, $\quad -\dfrac{1}{2\mu}\nabla^2\vec{u}_2 + \vec{u}_2\cdot\nabla\vec{u}_2 = \vec{f} + \tfrac{1}{4}\nabla\pi,$

and it is evident then, that for $\nu = \dfrac{1}{2\mu}$, $\left(\vec{u}_1, \tfrac{1}{4}\pi\right)$, $\left(\vec{u}_2, -\tfrac{1}{4}\pi\right)$ are two distinct solution pairs for (2).

The uniqueness theorem asserts that for $\nu > 0$ sufficiently large, the weak solution is unique. Here, suppose $\vec{\phi}$ is a weak solution of (1). Then

$$\frac{1}{\mu}\left[\left[\vec{\phi},\vec{\phi}\right]\right] = b\left(\vec{a},\vec{\phi},\vec{\phi}\right) + b\left(\vec{\phi},\vec{a},\vec{\phi}\right) = b\left(\vec{\phi},\vec{a},\vec{\phi}\right)$$

and $\qquad \dfrac{1}{\mu}\|\vec{\phi}\|_V^2 \le C\|\vec{a}\|_V\|\vec{\phi}\|_V^2.$

This leads to $\qquad (\dfrac{1}{\mu} - C\|\vec{a}\|_V)\|\vec{\phi}\|_V^2. \le 0$

which implies $\qquad \|\vec{\phi}\|_V = 0 \ \textit{if} \ \dfrac{1}{\mu} > C\|\vec{a}\|_V;$

Then (1) can only have a trivial solution if $2\nu > C\|\vec{a}\|_V$.

Inhomogeneous Boundary Data

Suppose $\vec{v}$ solves

$$\begin{aligned} -\nu\nabla^2\vec{v} + \vec{v}\cdot\nabla\vec{v} + \nabla p &= \vec{f} && \textit{in } U \\ \textit{div}\,\vec{v} &= 0 && \textit{in } U, \qquad (1) \\ \vec{v} &= \vec{g} && \textit{on } \partial U. \end{aligned}$$

We can reduce this to a problem with homogeneous data on ∂U by letting $\vec{u} = \vec{v} - \vec{V}$ where $\vec{V}$ denotes a smooth, divergence free vector field such that $\vec{V}|_{\partial U} = \vec{g}$. Then

$$-\nu\nabla^2\vec{u} + \vec{u}\cdot\nabla\vec{u} = -\nu\nabla^2\vec{v} + \nu\nabla^2\vec{V} + (\vec{v}-\vec{V})\cdot\nabla(\vec{v}-\vec{V})$$
$$= -\nu\nabla^2\vec{v} + \nu\nabla^2\vec{V} + \vec{v}\cdot\nabla\vec{v} + \vec{V}\cdot\nabla\vec{V} - \vec{V}\cdot\nabla\vec{v} - \vec{v}\cdot\nabla\vec{V}$$
$$= \vec{f} - \nabla p + \nu\nabla^2\vec{V} + \vec{V}\cdot\nabla\vec{V} - \vec{V}\cdot\nabla\vec{v} - \vec{v}\cdot\nabla\vec{V}$$

and

$$\nu[[\vec{u},\vec{u}]] + b(\vec{u},\vec{u},\vec{u}) = ((\vec{f},\vec{u}))_0 + \nu[[\vec{V},\vec{u}]] + b(\vec{V},\vec{V},\vec{u}) - b(\vec{V},\vec{v},\vec{u}) - b(\vec{v},\vec{V},\vec{u}).$$

Now, $\quad b(\vec{V},\vec{V},\vec{u}) - b(\vec{v},\vec{V},\vec{u}) = b(\vec{u},\vec{V},\vec{u})$

$$b(\vec{V},\vec{v},\vec{u}) = \underbrace{b(\vec{V},\vec{v},\vec{u}) - b(\vec{V},\vec{V},\vec{u})} + b(\vec{V},\vec{V},\vec{u})$$

$$. \qquad b(\vec{V},\vec{u},\vec{u}) = 0,$$

and so,

$$\nu[[\vec{u},\vec{u}]] + 0 = ((\vec{f},\vec{u}))_0 + \nu[[\vec{V},\vec{u}]] + b(\vec{u},\vec{V},\vec{u}) + b(\vec{V},\vec{V},\vec{u}).$$

Then we have

$$\left|((\vec{f},\vec{u}))_0 + \nu[[\vec{V},\vec{u}]] + b(\vec{V},\vec{V},\vec{u})\right| \le \left(\|\vec{f}\|_{V'} + \nu\|\vec{V}\|_V + C\|\vec{V}\|_V^2\right)\|\vec{u}\|_V$$

If the following condition

$$\left|b(\vec{u},\vec{V},\vec{u})\right| \le \alpha\|\vec{u}\|_V^2 \qquad \text{for some } \alpha < \nu \qquad (*)$$

is satisfied, then it follows that

$$(\nu-\alpha)\|\vec{u}\|_V^2 \le \left(\|\vec{f}\|_{V'} + \nu\|\vec{V}\|_V + C\|\vec{V}\|_V^2\right)\|\vec{u}\|_V$$

which leads immediately to the a-priori estimate

$$\|\vec{u}\|_V \le \frac{1}{(\nu-\alpha)}\left(\|\vec{f}\|_{V'} + \nu\|\vec{V}\|_V + C\|\vec{V}\|_V^2\right).$$

This estimate is the principal ingredient in proving the existence of a $\vec{u} \in V$ such that

$$\nu[[\vec{u},\vec{w}]] + b(\vec{u},\vec{u},\vec{w}) = ((\vec{f},\vec{w}))_0 + \nu[[\vec{V},\vec{w}]] - b(\vec{V},\vec{V},\vec{w}) - b(\vec{u},\vec{V},\vec{w}) \qquad \forall\vec{w} \in V$$

Evidently, existence for (1) depends on being able to extend $\vec{g}$ off of ∂U as a divergence free vector field satisfying $(*)$. We can view the condition $(*)$ as a lower bound for ν sufficient for existence, or, we can view $(*)$ as a condition on the set U and on $\vec{g}$; i.e., U and $\vec{g}$ must be such that for all $\alpha > 0$, there exists $\vec{V} = \vec{V}(\alpha)$, a divergence free extension of $\vec{g}$ such that $(*)$ holds. When ∂U has only one connected component, this reduces to a requirement on the smoothness of ∂U and $\vec{g}$. When ∂U has more than one connected component, it is more complicated to find the conditions on ∂U and $\vec{g}$ which imply $(*)$. Note that

$$((div\,\vec{u},1))_0 = (\vec{u},grad\,1)_0 - \langle T\vec{u},1\rangle = \int_U \vec{u}\bullet\nabla(1)dx - \int_{\partial U} 1\,T\vec{u}\,ds,$$

$$0 = -\int_{\partial U} T\vec{u}\,ds = -\int_{\partial U}\vec{g}\bullet\vec{n}\,ds.$$

Therefore the data $\vec{g}$ must satisfy the usual compatibility condition for any solution to exist.

Regularity of Solutions to Steady State N-S Equations

Using the same estimation techniques that were used for linear elliptic boundary value problems, the following regularity results can be proved.

1) **Interior Regularity** Suppose $U \subset R^n,\ n \ge 2,$ is any bounded open set. Suppose $\vec{v} \in W^{1,2}_{loc}(U)^n,\ p \in L^2_{loc}(U)$ are a weak solution pair for the stationary N-S equations. Then

$$f \in L^q_{loc}(U) \quad implies \quad \vec{v} \in W^{2,2}_{loc}(U)^n,\ p \in W^{1,2}_{loc}(U),$$

where

$$1 < q < \infty \quad if \quad n = 2$$

$$\frac{2n}{n+2} \leq q < \infty \quad if \quad n \geq 3.$$

Moreover, $f \in W^{m,q}_{loc}(U) \quad implies \quad \vec{v} \in W^{m+2,2}_{loc}(U)^n,\ p \in W^{m+1,2}_{loc}(U),$
where

$$1 < q < \infty \quad if \quad n = 2$$

$$\frac{n}{2} \leq q < \infty \quad if \quad n \geq 3.$$

2) **Regularity Up To The Boundary** Suppose $U \subset R^n,\ n \geq 2,$ is a bounded set whose boundary has at least C^2 regularity. Suppose $\vec{v} \in W^{1,2}(U)^n,\ p \in L^2(U)$ are a weak solution pair for the stationary N-S equations. Then

$$f \in L^q(U) \ and\ \vec{u}_* \in W^{2-1/q,q}(\partial U) \quad implies \quad \vec{v} \in W^{2,q}(U)^n,\ p \in W^{1,q}(U),$$

where

$$1 < q < \infty \quad if \quad n = 2$$

$$\frac{2n}{n+2} \leq q < \infty \quad if \quad n \geq 3.$$

Moreover, if $\partial U \in C^{m+2},\quad f \in W^{m,q}(U) \quad and\ \vec{u}_* \in W^{m+2-1/q,q}(\partial U)$

then $\vec{v} \in W^{m+2,2}(U)^n, \quad p \in W^{m+1,2}(U),$
where

$$1 < q < \infty \quad if \quad n = 2$$

$$\frac{n}{2} \leq q < \infty \quad if \quad n \geq 3.$$

Transition to the Stokes Problem as $\nu \to \infty$

The linear Stokes problem can be viewed as an approximation to the nonlinear stationary N-S equations when $\vec{u} \cdot \nabla\vec{u}$ is small compared to $-\nu\nabla^2\vec{u}.$ The following theorem shows the sense in which the transition from N-S to Stokes equations occurs as $\nu \to \infty.$

Transition Theorem

Suppose $U \subset R^n,\ n = 2,3,$ has Lipschitz smooth boundary, and suppose $\vec{v}_* \in H^{1/2}(\partial U),\ \vec{f} \in V'.$ Let $\vec{v} = \vec{v}(x;\nu),\ p = p(x,\nu)$ denote a solution pair for the steady state N-S equations corresponding to viscosity $\nu > 0$ and data $\vec{f},\ \vec{v}_*.$ Let $\vec{w}, \pi$ denote the solution pair for the associated linear Stokes problem with the same data. Then for any $\nu_0 > 0,$ there exists $C = C\left(\vec{f},\ \vec{v}_*, \nu_0\right) > 0$ such that

$$\|\vec{v} - \vec{w}\|_V + \|p - \pi\|_0 \leq \frac{C}{\nu} \qquad \forall \nu \geq \nu_0.$$

Proof- First, write

$$\left[\left[\vec{v}-\vec{w},\vec{\phi}\right]\right]+\frac{1}{\nu}b(\vec{v},\vec{v},\vec{\phi}) = 0 \qquad \forall\vec{\phi}\in V,$$

or, letting $\vec{u} = \vec{v}-\vec{w}$, and taking $\vec{\phi} = \vec{u}$

$$\|\vec{u}\|_V^2 = \left|-\frac{1}{\nu}\,b(\vec{v},\vec{v},\vec{u})\right| \le \frac{C}{\nu}\,\|\vec{v}\|_V^2\,\|\vec{u}\|_V.$$

Then $$\|\vec{u}\|_V \le \frac{C}{\nu}\,\|\vec{v}\|_V^2 \le \frac{C}{\nu}C_1^2\left(\frac{1}{\nu}\,\|\vec{f}\|_{V'}+\frac{k}{\nu}\,\|\vec{v}_*\|_{H^{1/2}(\partial U)}^2\right)^2$$

We also have

$$\nu\,[[\vec{u},\vec{\psi}]] + b(\vec{v},\vec{v},\vec{\psi}) = (p-\pi,\nabla\bullet\vec{\psi})_0 \qquad \forall\psi\in V,$$

which leads to

$$\|p-\pi\|_0 \le C\left(\nu\,\|\vec{u}\|_V+\|\vec{v}\|_V^2\right).$$

Combining this with the previous estimate leads to the conclusion.■