A Nonlinear Parabolic Initial Boundary Value Problem

For $U \subset R^n,$ open, bounded with smooth boundary $\Gamma,$ consider the IBVP

$$\begin{aligned} \partial_t u(x,t) - div(a(u(x,t))\nabla u(x,t)) &= f(x,t) \qquad in\ U \times (0,T) \\ u(x,t) &= 0 \qquad on\ \ \Gamma \times (0,T) \\ u(x,0) &= u_0(x) \quad in\ U \end{aligned}$$

where

$$a : R \to R \quad \text{belongs to } L^\infty(R)$$

and

$$0 < C_0 \le a(s) \le C_1 \quad \forall s \in R.$$

Let $\qquad V = H_0^1(U) \subset H = L^2(U) \subset V' = H^{-1}(U)$

and define $\qquad a(u,v) = \int_U a(u)\nabla u \bullet \nabla v\ dx \quad$ for $u,v \in V$

Evidently, $\qquad |a(u,v)| \le C_1|\int_U \nabla u \bullet \nabla v\ dx| \le C_1\|u\|_V\|v\|_V$

and it follows that $A(u)$ defined by

$$\langle A(u), v\rangle_{V'\times V} = a(u,v) \qquad \text{for } u,v \in V$$

defines a nonlinear map from V into V' such that for all $u \in V,$

$$\|A(u)\|_{V'} \le C_1\|u\|_V$$

Then we define $u(t) \in L^2[0,T : V]$ to be a weak solution of the IBVP if

$$(1) \qquad \left\{ \begin{array}{c} (u'(t),v)_H + a(u(t),v) = (f(t),v)_H \qquad forall\ v \in V, \\ u(0) = u_0 \end{array} \right\}$$

Alternatively, $u(t)$ must satisfy

$$(2) \qquad u'(t) + A(u(t)) = f(t), \qquad u(0) = u_0.$$

Existence and Uniqueness Theorem

Suppose $f \in L^2[0,T : H]$ and $u_0 \in H$. Then there exists a unique weak solution for the IBVP. This weak solution has the following additional smoothness,

$$u'(t) \in L^2[0,T : V'] \quad and \qquad u(t) \in C[0,T : H]$$

Proof- Since the embedding of V in H is compact, it follows that there is an orthonormal basis for H, $\{w_k\}$, which is simultaneously an orthogonal basis for V; i.e.,

$$(w_j, w_k)_H = \left\{ \begin{array}{ll} 0 & if \quad j \neq k \\ \lambda_j (w_j, w_j)_V & if \quad j = k \end{array} \right\}$$

Then, for $N = 1,2,...$ let

$$u_N(t) = \sum_{k=1}^{N} c_{k,N}(t)\, w_k$$

satisfy,

$$(3) \quad \left\{ \begin{array}{c} (u_N'(t), w_j)_H + a(u_N(t), w_j) = (f(t), w_j)_H \qquad for\, j = 1,...,N \\ (u_N(0) - u_0, w_j)_H = 0 \qquad for\, j = 1,...,N \end{array} \right\}$$

This is a system of nonlinear ODE's for the coefficients $\{c_{j,N}(t)\}$,

$$\begin{array}{l} c_{j,N}'(t) + \sum_{k=1}^{N} B_{j,k}(c_1,...,c_N)\, c_{k,N}(t) \; = f_j(t) \\ \qquad c_{j,N}(0) = (u_0,, w_j)_H \end{array}$$

where

$$B_{j,k}(c_1,...,c_N) = \int_U a(u_N) \nabla w_j \bullet \nabla w_k \; dx$$

For each fixed N, it is well known that this nonlinear IVP has a unique solution on an interval $[0, T_N]$ with $T_N \leq T$. In order to have $T_N < T$ the solution would have to become infinite as t tends to T$_N$. However, the a-priori estimates we are about to prove will show that such unbounded behavior for $u_N(t)$ is not possible. Therefore, for each N, an approximate solution $u_N(t)$ exists on $[0,T]$.

a-priori estimates

It follows from (3) that

$$(4) \qquad \tfrac{d}{dt} \|u_N(t)\|_H^2 + 2a(u_N(t), u_N(t)) = 2(f(t), u_N(t))_H.$$

Then,

$$\tfrac{d}{dt} \|u_N(t)\|_H^2 + 2C_0 \|u_N(t)\|_V^2 \leq 2\|f(t)\|_H \|u_N(t)\|_H$$

$$\leq (1/C_0)\,\|f(t)\|_H^2 + C_0\,\|u_N(t)\|_V^2$$

or,

(5) $$\tfrac{d}{dt}\|u_N(t)\|_H^2 + C_0\,\|u_N(t)\|_V^2 \leq (1/C_0)\,\|f(t)\|_H^2$$

Then, clearly,

$$\tfrac{d}{dt}\|u_N(t)\|_H^2 \leq (1/C_0)\,\|f(t)\|_H^2$$

Integrate this last expression from $t = 0$ to $t = \tau$, to get

$$\|u_N(\tau)\|_H^2 \leq \|u_N(0)\|_H^2 + (1/C_0)\int_0^T \|f(t)\|_H^2\,dt \leq \|u_0\|_H^2 + \|f\|_{L^2[0,T,H]}^2$$

Since this holds for all $\tau \in [0,T]$ it follows that

(6) $$\|u_N\|_{L^\infty[0,T:H]}^2 \leq \|u_0\|_H^2 + \|f\|_{L^2[0,T,H]}^2 = M_1$$

This shows that the sequence of approximate solutions is a bounded infinite set in $L^\infty[0,T : H]$

Integrating (5) from 0 to T, leads to

$$\|u_N(T)\|_H^2 + C_0\int_0^T \|u_N(t)\|_V^2\,dt \leq \|u_N(0)\|_H^2 + (1/C_0)\int_0^T \|f(t)\|_H^2\,dt$$

i.e.,

(7) $$\|u_N\|_{L^2[0,T:V]}^2 \leq (1/C_0)\,\|u_0\|_H^2 + (1/C_0)^2\,\|f\|_{L^2[0,T:H]}^2 = M_2$$

This shows that the sequence of approximate solutions is a bounded infinite set in $L^2[0,T : V]$.

Next, for each N, let $P_N : V \to V_N = span\{w_1,...,w_N\}$ denote the projection from V into V$_N$. Then it follows from (3) that for each N,

$$\langle u_N'(t) + A(u_N(t)) - f(t), P_N v\rangle_{V'\times V} = 0 \qquad \forall v \in V$$

This is equivalent to

$$\langle P_N^\top\{u_N'(t) + A(u_N(t)) - f(t)\}, v\rangle_{V'\times V} = 0 \qquad \forall v \in V$$

where $P_N^\top : V_N' \to V'$ But $P_N^\top\{u_N'(t) = u_N'(t)$, so

$$\|u_N'(t)\|_{L^2[0,T:V']} = \left(\int_0^T (\sup_{\|v\|_V=1} |\langle u_N'(t), v\rangle_{V'\times V}|)^2\,dt\right)^{1/2}$$

$$\|u_N'(t)\|_{L^2[0,T:V']} = \|P_N^{\top}\{A(u_N(t)) - f(t)\}\|_{L^2[0,T:V']}$$

$$\leq \|A(u_N(t)) - f(t)\|_{L^2[0,T:V']}$$

$$\leq \|A(u_N(\bullet))\|_{L^2[0,T:V']} + \|f\|_{L^2[0,T:V']}$$

$$\leq C_1\|u_N\|_{L^2[0,T:V]} + \|f\|_{L^2[0,T:V']}; \quad i.e.,$$

(8) $$\|u_N'(t)\|_{L^2[0,T:V']} \leq M_1 C_1 + \|f\|_{L^2[0,T:V']} = M_3$$

This shows that the sequence of derivatives approximate solutions is a bounded infinite set in $L^2[0,T : V']$

passing to the limit
The a-priori estimates (7) and (8) imply the existence of a subsequence of $\{u_N(t)\}$ such that

(a) $u_N(t) \to u(t)$ weakly in $L^2[0,T : V]$

(b) $u_N'(t) \to v(t)$ weakly in $L^2[0,T : V']$

The usual distributional argument is used to show that $v(t) = u'(t)$.
It follows from (a) and the fact that V is compactly embedded in H that there is a further subsequence (still denoted by $\{u_N(t)\}$) such that

(c) $u_N(t) \to u(t)$ strongly in $L^2[0,T : H]$

It also follows from (7) and the assumptions on $a(\bullet)$, that

$$\|A(u_N(\bullet))\|_{L^2[0,T:V']} \leq C_1 M_2$$

which leads to

(d) $A(u_N(\bullet)) \to B_1$ weakly in $L^2[0,T : V']$

Now let

$$b(u) = \int_0^u a(s)\ ds;$$

i.e., $b'(u) = a(u)$, $b(0) = 0$, and $b(u)$ is continuous on R.
Also,

$$C_0\,|u| \leq |b(u)| \leq C_1\,|u| \qquad \forall u \in R$$

Since $\nabla b(u) = a(u)\nabla u,$ it follows from (6) and (7) that

$$\|b(u_N(\bullet))\|_{L^2[0,T:V]} \leq C_1 M_1 \qquad \forall N.$$

It now follows that

(e) $b(u_N(\bullet)) \to B_2$ weakly in $L^2[0,T:V]$

and the compactness of the embedding $V \subset H$ then implies that $b(u_N(\bullet)) \to B_2$ strongly in $L^2[0,T:H]$. But $b(\bullet)$ is continuous on R and $u_N(t) \to u(t)$ strongly in $L^2[0,T:H]$. Then it follows that $b(u_N(\bullet)) \to b(u)$ strongly in $L^2[0,T:H]$, hence $b(u) = B_2$.

Now $$a(u,v) = \int_U a(u)\nabla u \bullet \nabla v \, dx = \int_U \nabla b(u) \bullet \nabla v \, dx$$

$$= (b(u),v)_V - (b(u),v)_H$$

hence for arbitrary $v \in V$

$$\int_0^T \langle A(u_N(t)),v\rangle_{V'\times V} \, dt = \int_0^T \{(b(u_N),v)_V - (b(u_N),v)_H\} \, dt.$$

Then we have

$$\int_0^T \{(b(u_N),v)_V - (b(u_N),v)_H\} \, dt \to \int_0^T \{(b(u),v)_V - (b(u),v)_H\} \, dt$$

and $$\int_0^T \langle A(u_N(t)),v\rangle_{V'\times V} \, dt \to \int_0^T \langle B_1,v\rangle_{V'\times V} \, dt.$$

Also $$\int_0^T \{(b(u),v)_V - (b(u),v)_H\} \, dt = \int_0^T a(u,v) \, dt = \int_0^T \langle A(u(t)),v\rangle_{V'\times V} \, dt$$

and together these results imply that $A(u(t)) = B_1; i.e., \; A(u_N(\bullet)) \to A(u(t)),$ weakly in $L^2[0,T:V']$. Then we can pass to the limit in (3) to see that $u(t)$ is a weak solution of the partial differential equation. Then $u'(t) = f(t) - A(u(t))$ belongs to $L^2[0,T:V']$ and it follows that $u(t) \in L^2[0,T:V] \cap C[0,T:H]$. In addition, we can show that $u(0) = u_0$.

To prove uniqueness, suppose that $u_1(t), u_2(t) \in L^2[0,T:V]$ are two weak solutions of (1). Then $w(t) = u_1(t) - u_2(t)$ solves

$$\langle \partial_t w(t),v\rangle_{V'\times V} + \langle A(u_1) - A(u_2),v\rangle_{V'\times V} = 0 \qquad \forall v \in V, \qquad w(0) = 0.$$

But $\langle A(u_1) - A(u_2), v\rangle_{V'\times V} = \int_U \nabla(b(u_1) - b(u_2)) \bullet \nabla v\, dx$

$$= (b(u_1) - b(u_2), v)_V - (b(u_1) - b(u_2), v)_H$$

and $(b(u_1) - b(u_2), v)_V = \langle b(u_1) - b(u_2), Jv\rangle_{V\times V'} = (b(u_1) - b(u_2), Jv)_H$

where J denotes the isomorphism that associates $v \in V$ with a unique element $Jv \in V'$ as prescribed by the fact that H is the pivot space between V and V' If we choose $v \in V$ such that $Jv = w,$ then

$$(b(u_1) - b(u_2), v)_V = (b(u_1) - b(u_2), w)_H$$

and $\langle A(u_1) - A(u_2), v\rangle_{V'\times V} = (b(u_1) - b(u_2), w)_H - (b(u_1) - b(u_2), v)_H.$

It follows that

$$\langle \partial_t w(t), v\rangle_{V'\times V} + (b(u_1) - b(u_2), w)_H = (b(u_1) - b(u_2), v)_H.$$

and

$$\langle \partial_t w(t), J^{-1} w\rangle_{V'\times V} + C_0 \|w\|_H^2 \leq C_1 \int_U |w(x)||J^{-1} w(x)|\ dx.$$

$$\leq \tfrac{1}{2} C_0 \|w\|_H^2 + C_2 \|J^{-1} w(x)\|_H^2.$$

Note also, that $\langle \partial_t w(t), J^{-1} w\rangle_{V'\times V} = (\partial_t w(t), w)_{V'} = \frac{1}{2}\frac{d}{dt}\|w(t)\|_{V'}^2$

and $\|J^{-1} w(x)\|_H^2 = (J^{-1} w(x), J^{-1} w(x))_H = (w, w)_{V'}.$

Then from $\frac{d}{dt}\|w(t)\|_{V'}^2 \leq 2C_2 \|w(t)\|_{V'}^2 \qquad w(0) = 0,$

it follows that $\|w(t)\|_{V'} = 0$ so the solution is unique.