Linear Operators
For reference purposes, we will collect a number of useful results regarding bounded and unbounded linear operators.

Bounded Linear Operators
Suppose $T$ is a bounded linear operator on a Hilbert space $H$. In this case we may suppose that the domain of $T$, $D(T)$, is all of $H$. For suppose it is not. Then let $D(T)^{CL}$ denote the closure of $D(T)$, and extend $T$ to the closure by continuity. That is, if $x \in D(T)^{CL}$ then there exists a sequence $\{x_n\} \subset D(T)$ such that $x_n \to x$ in $H$. Since $T$ is continuous, $Tx_n$ converges to some limit $f \in H$ and we can define $Tx = f$. It is not difficult to show that the value $f$ depends on $x$ but does not depend on the sequence $\{x_n\}$. If $D(T)^{CL}$ is not all of $H$, then it is a closed subspace of $H$ and, as such, has an orthogonal complement in $H$. Then extend $T$ by zero to this orthogonal complement to obtain a linear operator which is now defined on all of $H$. It is easy to show that this extended operator is still continuous and has the same norm as $T$.

Recall that the linear operator $T$ is said to be well defined on $H$ if $Tv = f$, and $Tv = g$ iff $f = g$. This is equivalent to the statement that there is no nonzero element $f$ such that $T0 = f$. The linear operator $T$ is said to be one to one on $H$ if $Tv = f$, and $Tu = f$ iff $u = v$. This is equivalent to the statement that $Tu = 0$ iff $u = 0$, (only the zero element is mapped to zero).

Adjoint of a Bounded Linear Operator
For $T$ a bounded linear operator on Hilbert space $H$ and a fixed $v$ in $H$, let

$$F(u) = (Tu, v)_H \quad \forall u \in H.$$ 

Clearly this defines a bounded linear functional $F$ on $H$ and it follows then from the Riesz theorem that there exists a unique element $z_F \in H$ such that

$$(Tu, v)_H = (u, z_F)_H \quad \forall u \in H.$$ 

If the correspondence relating $z_F$ to $v$ is denoted by $z_F = T^*v$, then

$$(Tu, v)_H = (u, T^*v)_H \quad \forall u \in H.$$ 

Then it is not hard to show $T^*$ is well defined on $H$ with values in $H$ and, moreover, $T^*$ is linear and bounded with $\|T^*\| = \|T\|$. For example, if $T^*v = f$, and $T^*v = g$ then

$$(Tu, v)_H = (u, T^*v)_H = (u, f)_H \quad \text{and} \quad (Tu, v)_H = (u, T^*v)_H = (u, g)_H,$$

hence

$$(u, f - g)_H = 0 \quad \forall u \in D(T) = H.$$ 

Then $f = g$ and $T^*$ is well defined. The operator $T^*$ is called the adjoint of $T$ and we have seen it is a well defined and bounded linear operator given only that $T$ is bounded. If, in addition, $T$ is onto, then the adjoint is one to one. To see this suppose $T$ is onto and $T^*v = 0$ for some $v$ in $H$. Then for arbitrary $f$ in $H$, there is some $u$ in $H$ such that $Tu = f$ and

$$(f, v)_H = (Tu, v)_H = (u, T^*v)_H = 0;$$

i.e. $v \perp H$ which implies $v$ is zero and so $T^*$ is one to one. Conversely, suppose $T^*$ is one to
one and \( f \perp \operatorname{Rng} T \). That is,
\[
0 = (Tu,f)_H = (u,T^*f)_H \quad \forall u \in H
\]
But this implies \( T^*f \perp H \), or \( T^*f = 0 \). But we assumed \( T^* \) is one to one and it follows that \( f = 0 \). Then \( f \perp \operatorname{Rng} T \) implies \( f = 0 \) which is to say, \( T \) is onto.

A bounded linear operator \( T \) such that \( (Tu,v)_H = (u,Tv)_H \) for all \( u,v \in H \) is said to be self adjoint. We are going to be interested in differential operators which are self adjoint but differential operators are typically not bounded from \( H \) into itself.

**Unbounded Linear Operators**

Consider the operator

\[
Tu(x) = u'(x), \quad \text{for} \ u \in D(T) = C^1[0,1] \subset L^2[0,1] = H.
\]

Then \( D(T) \) is dense in \( H \) but it is not all of \( H \). Note that \( \{\phi_n(x)\} = \{\sin(n\pi x)\} \) is a sequence in \( D(T) \) that is bounded in \( H \) but the sequence \( \{T\phi_n(x)\} = \{(n\pi) \cos(n\pi x)\} \) is an unbounded sequence in \( H \). Then \( T \) is densely defined and unbounded on \( H \). Note that if we took \( H \) to be the Hilbert space \( H^1[0,1] \) then \( T \) is defined on all of \( H \) but \( T \) does not map \( H \) into \( H \). Note further that if we define

\[
Su(x) = u'(x), \quad \text{for} \ u \in D(S) = H^1[0,1] \subset L^2[0,1] = H,
\]

then \( D(T) \subset D(S) \subset H \) and \( Tu = Su \) for all \( u \in D(T) \). Then \( T \) is a restriction of \( S \) and \( S \) is an extension of \( T \) and both of these operators are densely defined in \( H \). When dealing with differential operators on a Hilbert space of functions, we will generally try to choose the largest possible domain for the operator. However, as we shall see, this is not always easy to do.

**Adjoint of an Unbounded Linear Operator**

Suppose \( T \) is an unbounded but densely defined linear operator on \( H \). Then we define \( y \in D(T^*) \subset H \) with \( T^*y = v \) to mean

\[
(Tu,v)_H = (u,v)_H \quad \forall u \in D(T).
\]

\( T^* \) is well defined because \( T \) is densely defined. To see this, suppose \( T^*y = v \) and \( T^*y = w \). Then

\[
(Tu,v)_H = (u,v)_H = (u,w)_H \quad \forall u \in D(T).
\]

But in this case \( (u,v-w)_H = 0 \) \( \forall u \in D(T) \), and since \( D(T) \) is dense in \( H \), this implies \( v = w \). It is easy to show that \( D(T^*) \) is a nonempty subspace of \( H \) (i.e. \( D(T^*) \) contains at least the zero element) and \( T^* \) is a linear operator from \( D(T^*) \) into \( H \).

For example, consider

\[
Tu(x) = u'(x),
\]

for \( u \in D(T) = \{u \in C^1[0,1] : u(0) = u(1) = 0\} \subset L^2[0,1] = H.\)
Then
\[(Tu, v)_H = \int_0^1 u'(x)v(x)dx = u(x)v(x)|_{x=1}^1 - \int_0^1 u(x)v'(x)dx = (u, -v')_H\]

Since this string of equalities is valid for any \(v\) in \(D(T) = C^1[0, 1]\), it follows that \(D(T^*)\) contains \(D(T)\), and moreover, \(T^*v = -v'(x)\). In fact, we can show that \(D(T^*) = C^1[0, 1]\) (with no boundary conditions).

On the other hand, consider
\[Tu(x) = u'(x),\]
for \(u \in D(T) = \{u, u' \in L^2[0, 1] : u(0) = 0\}\).

Then \(y \in D(T^*) \subset H\) with \(T^*y = v\) means
\[(Tu, y)_H = (u, v)_H \quad \forall u \in D(T).\]

Let \(F(x) = \int_1^x v,\) so that \(F' = v\) and \(F(1) = 0\).

Then
\[(u, v)_H = \int_0^1 u(x)F'(x)dx = Fu|_{x=0}^1 - \int_0^1 u'(x)F(x)dx = -(Tu, F)_H\]
and
\[(Tu, y + F)_H = 0 \quad \forall u \in D(T).\]

Now choose \(u_0(x) = \int_0^x [y(s) + F(s)]ds,\)
which implies
\(u_0 \in L^2[0, 1],\quad u_0' = y + F \in L^2[0, 1],\) and \(u_0(0) = 0;\)
i.e., \(u_0 \in D(T).\) Moreover, then \((Tu_0, y + F)_H = \|y + F\|^2_H = 0,\)
which leads to
\[y(x) = -F(x) = -\int_1^x v,\]
or
\[-y'(x) = v(x) = T^*y,\quad y, y' \in L^2[0, 1],\quad y(1) = 0.\]

We can deduce from this that \(T^*y = -y'(x),\) \(D(T^*) = \{y, y' \in L^2[0, 1],\quad y(1) = 0\},\)

A densely defined linear operator \(T\) on \(H\) is said to be symmetric if
\[(Tu, v)_H = (u,Tv)_H \quad \forall u, v \in D(T).\]

In this case \(D(T) \subset D(T^*)\) and \(T^*u = Tu \forall u \in D(T).\) Then \(T^*\) is an extension of \(T.\) If it is the case that \(T\) is symmetric and, in addition, \(D(T) \subset D(T^*),\) then we say \(T\) is self-adjoint.
For any bounded, symmetric operator $T$, $D(T) = D(T^*) = H$ so the operator is then self adjoint; i.e., for bounded operators symmetry and self adjointness are equivalent. For unbounded operators, they are in general different. For example
\[ T_1u(x) = u''(x) \quad \text{with} \quad D(T_1) = \{u \in C^2[0,1] : u(0) = u(1) = 0\}. \]
can be easily shown to be self adjoint, in contrast to
\[ T_2u(x) = u''(x) \quad \text{with} \quad D(T_2) = \{u \in C^2[0,1] : u(0) = u(1) = 0, u'(0) = u'(1) = 0\}. \]
which is symmetric but not selfadjoint. Similarly,
\[ T_3u(x) = iu'(x) \quad \text{with} \quad D(T_3) = \{u \in C^1[0,1] : u(0) = u(1) = 0\} \]
is symmetric but not selfadjoint while
\[ T_4u(x) = iu'(x) \quad \text{with} \quad D(T_4) = \{u \in C^1[0,1] : u(0) = 0\} \]
is neither symmetric nor selfadjoint.

**Closed Linear Operators**
Recall that a linear operator $T$ on $H$ is said to be bounded if there exists a constant $C > 0$ such that $\|Tx\|_H \leq C\|x\|_H$ for all $x$ in $H$. $T$ is said to be continuous if $x_n \to x$ in $H$ implies $Tx_n \to Tx$ in $H$. For linear operators, these two notions are equivalent. A densely defined linear operator $T$ is said to be closed if
\[
\begin{align*}
  x_n &\in D(T) \\
x_n &\to x \text{ in } H \\
T x_n &\to f \text{ in } H
\end{align*}
\]
implies
\[
\begin{align*}
x &\in D(T) \\
Tx &= f
\end{align*}
\]
Note that $T$ continuous implies $T$ closed but not conversely. Differential operators are not continuous but are closed, (if the domain is properly chosen). We have seen that if $T$ is linear and densely defined, then the adjoint $T^*$ is well defined. In addition, $T^*$ is closed, whether or not $T$ itself is closed. To see this, suppose $x_n \in D(T^*), \; x_n \to x \text{ in } H \text{ and } T^* x_n \to f \text{ in } H$. Then for any $z \in D(T)$,
\[
(Tz, x_n)_H = (z, T^* x_n)_H, \quad (Tz, x_n)_H \to (Tz, x)_H, \text{ and } (z, T^* x_n)_H \to (z, f)_H,
\]
which is to say, $(Tz, x)_H = (z, f)_H \; \forall z \in D(T)$. But this implies $x \in D(T^*)$ and $T^* x = f$, so $T^*$ is closed. Note that it was not necessary to suppose that $T$ was closed.

We have seen in previous examples that it may be possible to specify the domain of a differential operator in several different ways. For example, $Tu(x) = u'(x)$ could have any of the following subspaces of $L^2[0,1]$ as its domain,
\[ D_1 = C^\infty[0,1] \subset D_2 = C^1[0,1] \subset D_3 = H^1[0,1]. \]

We would like to choose the domain as large as possible, and moreover to have \( T \) on this domain be closed. To see how this might be done, recall that if \( T \) is linear and densely defined then the adjoint \( T^* \) is linear and well defined. Now if \( T^* \) is also densely defined, then the adjoint of \( T^* \), denoted \( T^{**} \), is also well defined and it is closed since it is an adjoint. More precisely, \( z \in D(T^{**}) \) with \( T^{**}z = w \) means
\[
(z, T^*y)_H = (w, y)_H = (T^{**}z, y)_H \quad \forall y \in D(T^*).
\]

But, for any \( y \in D(T^*) \), \( (z, T^*y)_H = (Tz, y)_H \quad \forall z \in D(T) \). This implies that \( D(T) \subset D(T^{**}) \) and \( T^{**}z = Tz \quad \forall z \in D(T) \); i.e., \( T^{**} \) is an extension of \( T \), and it is closed. Thus if the densely defined operator \( T \) is not closed but its adjoint \( T^* \) is densely defined, then \( T \) has a closed extension which is just the adjoint of \( T^* \). The domain of this extension must contain the limit points of all sequences \( \{u_n\} \) in \( D(T) \) for which the sequence \( \{Tu_n\} \) also converges. This is not the same thing as the closure of \( D(T) \) as the following theorem shows.

**Theorem (Closed Graph Theorem)** A closed operator on a closed domain is necessarily bounded.

The theorem shows that if \( D(T^{**}) \) were equal to \( D(T)^{CL} \) then the closed operator \( T^{**} \) on the closed domain \( D(T)^{CL} \) would be necessarily bounded. But this would contradict the fact that \( T^{**}z = Tz \quad \forall z \in D(T) \) and \( T \) is not bounded. Then, what we have instead is the following,
\[
D(T) \subset D(T^{**}) \subset D(T)^{CL} = H,
\]
where \( T \) is extended to the larger domain \( D(T^{**}) \) which is also dense in \( H \) but is not closed. In summary, we can say that if \( T \) is a densely defined linear operator on \( H \) then

a) if \( T^* \) is also densely defined then \( T^{**} \) is well defined and \( T^{**} \) is a closed extension of \( T \)

b) if \( T \) is closed then \( T^* \) is densely defined and \( T = T^{**} \)

**Solvability of Linear Equations**
Let \( A \) denote a densely defined linear operator on \( H \), and let
\[
R(A) = \{ y \in H : y = Ax \text{ for some } x \in D(A) \}, \\
N(A) = \{ x \in D(A) : Ax = 0 \}.
\]

Since \( A \) is densely defined, \( A^* \) is well defined and we will let
\[
R(A^*) = \{ y \in H : y = A^*x \text{ for some } x \in D(A^*) \}.
\]
\[ N(A^*) = \{ x \in D(A^*) : A^*x = 0 \} \]

Then we have the following facts relating these subspaces:

1. \( R(A)^{\perp} = N(A^*) \)

2. \((R(A)^{\perp})^{\perp} = R(A)^{\text{CL}} = N(A^*)^{\perp}\)

To see that 1 is true, suppose first that \( z \in R(A)^{\perp} \). Then \( (Au, z)_H = 0 \ \forall u \in D(A) \). Then \( z \in D(A^*) \), with \( A^*z = 0 \) since

\[ 0 = (Au, z)_H = (u, A^*z)_H \ \forall u \in D(A). \]

This proves that \( R(A)^{\perp} \subset N(A^*) \). now suppose \( z \in N(A^*) \), i.e., \( (x, A^*z)_H = 0 \ \forall x \in H \). In particular, if \( x \in D(A) \) then

\[ 0 = (x, A^*z)_H = (Ax, z)_H \ \forall x \in D(A). \]

But this implies \( z \perp R(A) \) and we have proved that \( N(A^*) \subset R(A)^{\perp} \). The two inclusions together imply 1. The proof of 2 is left as an exercise. Note that 2 implies that if \( A \) has closed range, then the equation \( Au = f \) is solvable if and only if \( f \perp N(A^*) \). In fact, we can show that

\[ R(A) = R(A)^{\text{CL}} \text{ if and only if } R(A^*) = R(A^*)^{\text{CL}} \]

and if this is the case, then

\[ R(A) = N(A^*)^{\perp} \text{ and } R(A^*) = N(A)^{\perp} \]

**Lemma 1** Suppose \( A \) is closed and densely defined, and for some \( C > 0 \),

\[ ||Au||_H \geq C||u||_H \ \forall u \in D(A). \]

Then \( A \) has closed range.

Proof- It is an exercise to show that the null space of any closed linear operator is a closed subspace of \( H \). Then for \( A \) a closed linear operator, we have \( H = N(A) \oplus N(A)^{\perp} \). Now let \( \{ v_n \} \subset D(A) \) be such that \( Av_n = f_n \to f \text{ in } H \); i.e., \( f \) is a limit point of \( R(A) \).

Now \( v_n \) can be written uniquely in the form

\[ v_n = u_n + w_n \ \text{ with } u_n \in N(A)^{\perp} \text{ and } w_n \in N(A). \]

Then for each \( n \),

\[ Av_n = Au_n \]
and
\[ \|f_n - f_m\|_H = \|Au_n - Au_m\|_H \geq C\|u_n - u_m\|_H. \]

But then \( \{u_n\} \) is a Cauchy sequence in \( H \) with limit point \( u \in H \), and since \( A \) is closed we have
\[
\begin{align*}
\left\{ \begin{array}{l}
Au_n \to f \\
u_n \to u
\end{array} \right\} & \quad \text{implies} \quad \left\{ \begin{array}{l}
u \in D(A) \\
Au = f
\end{array} \right\}.
\end{align*}
\]

Then \( f \in R(A) \) and it follows that \( A \) has closed range.\( \blacksquare \)

**Corollary**- Suppose \( A \) is closed and densely defined, and for some \( C > 0 \),
\[
(Au,u)_H \geq C\|u\|_H^2 \quad \forall u \in D(A).
\]

Then \( A \) has closed range.

An operator \( A \) with the property of the corollary is said to be accretive (and \(-A\) is said to be dissipative).

**Lemma 2**- Suppose \( A \) is densely defined and self adjoint, and for some \( C > 0 \),
\[
(Au,u)_H \geq C\|u\|_H^2 \quad \forall u \in D(A) \cap N(A^\perp).
\]

Then precisely one of the following alternatives must hold:

a) \( Au = f \) has a unique solution for every \( f \in H \)

b) \( Au = f \) has no solution unless \( f \perp N(A^\perp) \).
   If \( f \perp N(A^\perp) \) then solutions exist but are not unique.

It is evident why this is true. The hypotheses imply that \( R(A) \) is closed, hence \( R(A) = N(A^*)^\perp \). Then \( R(A) \) is either all of \( H \) or else the orthogonal complement of \( R(A) \) is just \( N(A^*) \). In addition, since \( A \) is self adjoint, \( N(A) = N(A^*) \) so the following statements are all equivalent, \( N(A^*) = \{0\}, \ R(A) = H, \ N(A) = \{0\} \).

**Invertibility of \( A \)**

For an unbounded operator \( A \), we define the inverse operator, \( A^{-1} \), by
\[
f \in D(A^{-1}) \text{ with } A^{-1}f = u \quad \text{if and only if} \quad u \in D(A) \text{ with } Au = f.
\]
Then for $u, v \in D(A)$, \[
\begin{align*}
A^{-1}f &= u \\
A^{-1}f &= v
\end{align*}
\Rightarrow \begin{cases} 
Au = f \\
Av = f
\end{cases} \Rightarrow A(u - v) = 0
\]
from which it is clear that $A^{-1}$ is well defined if and only if $A$ is one to one.

In particular, if $A$ is densely defined and if there exists $C > 0$ such that
\[
(Au, u)_H \geq C\|u\|_H^2 \quad \forall u \in D(A).
\]
then $A^{-1}$ exists. In addition, in this case $A^{-1}$ is necessarily bounded. To see this note that if $A^{-1}$ were not bounded then we could find a sequence $\{u_n\} \subset D(A)$ with
\[
\|u_n\|_H = 1 \quad \text{and} \quad \|Au_n\|_H \to 0.
\]
This would contradict the accretiveness assumption.

If $A$ is one to one and closed, then $A^{-1}$ is well defined and is also closed. To see this, suppose $\{f_n\} \subset R(A)$ is such that $f_n \to f$, and $A^{-1}f_n \to u$. To show $A^{-1}$ is closed we have to show $f \in R(A)$ and $A^{-1}f = u$. Then let $u_n = A^{-1}f_n$ and note that since $A$ is closed, we have that
\[
\begin{cases}
\quad u_n \to u, \\
\quad Au_n \to f
\end{cases} \quad \text{implies} \quad \begin{cases}
\quad u \in D(A) \\
\quad Au = f
\end{cases}
\]
But this means $f \in R(A)$ and $A^{-1}f = u$ and we are done.

If $A$ is one to one and closed, then $R(A)$ is closed if and only if $A^{-1}$ is bounded. To see why this is true suppose first that $A^{-1}$ is bounded and let $\{f_n\}$ a sequence in $R(A)$ with $f_n \to f$. Since $A^{-1}$ is bounded, $A^{-1}f_n = u_n$ is a Cauchy sequence in $H$ with limit $u$. Then, since $A$ is closed,
\[
\begin{cases}
\quad u_n \to u, \\
\quad Au_n = f_n \to f
\end{cases} \quad \text{implies} \quad \begin{cases}
\quad u \in D(A) \\
\quad Au = f
\end{cases}
\]
Then $f \in R(A)$ and it follows that $R(A)$ is closed. Conversely, if $R(A)$ is closed then it follows from the closed graph theorem that $A^{-1}$ is bounded (since $A^{-1}$ is known to be closed).

If $A$ is closed and densely defined on $H$ then
\[
R(A) = H \quad \text{iff} \quad (A^*)^{-1} \text{ is bounded}
\]
and
\[
R(A^*) = H \quad \text{iff} \quad (A)^{-1} \text{ is bounded}.
\]
To see this, suppose first $R(A) = H$. Since $(Ax, y)_H = (x, A^*y)_H \quad \forall x \in D(A)$, we conclude that $A^*y = 0$ \iff $y = 0$. Then $(A^*)^{-1}$ is well defined, and since $A^*$ is closed, $(A^*)^{-1}$ is closed. But $R(A) = H$ is closed so $R(A^*)$ is also closed, and then, by the closed graph theorem, $(A^*)^{-1}$ is bounded. Conversely, suppose $(A^*)^{-1}$ is bounded. Then $N(A^*) = \{0\}$ and it follows that $R(A^*)$ is closed. But then $R(A)$ is closed and equal to $N(A^*)^{-1} = H$.

**Spectral Theory for Closed Operators**

Let $A$ be closed and densely defined on $H$. Then for every complex number $\lambda$, $\lambda I - A$ is also closed and has the same domain as $A$. We define the following terms
\( \lambda \) belongs to the **resolvent set** for \( A \) if \((\lambda I - A)^{-1}\) exists and is bounded with 
\[ R(\lambda I - A) = H \]

\( \lambda \) belongs to the **point spectrum** for \( A \) if \((\lambda I - A)^{-1}\) fails to exist; 
i.e., \( \exists u \neq 0 \) such that \((\lambda I - A)u = H = 0\). Then \( \lambda \) is an eigenvalue 
for \( A \) and \( x \in N(\lambda I - A) \) is an eigenvector for \( A \)

\( \lambda \) belongs to the **continuous spectrum** for \( A \) if \((\lambda I - A)^{-1}\) exists but is not bounded 
with 
\[ R(\lambda I - A) \neq H \text{ but } R(\lambda I - A)^{CL} = H \]

\( \lambda \) belongs to the **residual or compression spectrum** for \( A \) if 
\[ R(\lambda I - A)^{CL} \neq H \]

We collect here several facts about the resolvent set and spectrum of \( A \). 
If \( A \) is closed, densely defined and symmetric then essentially the same proofs that 
work in linear algebra for symmetric matrices can be used to show that

\((Au, u)\)_\(H\) is real for all \( u \) in \( D(A) \)

all eigenvalues of \( A \) are real 
eigenvectors corresponding to distinct eigenvalues are orthogonal;
i.e., \( \lambda \neq \mu \) implies \( N(\lambda I - A) \perp N(\mu I - A) \).

If, in addition, \( A \) is self adjoint on \( H \), then the resolvent set of \( A \) contains the complement of 
the real axis. Moreover, for \( \text{Im}(\lambda) \neq 0 \), \((\lambda I - A)^{-1}\) is bounded and satisfies

\[ \| (\lambda I - A)^{-1} \|_{L(H)} \leq \frac{1}{|\text{Im}(\lambda)|} \]

and

\[ \text{Im}((A - \lambda I)x, x)_H = \text{Im}(\lambda) \| x \|_H^2 \quad \forall x \in D(A). \]

This can be seen by writing

\[ \|(A - \lambda I)x\|_H \| x \|_H \geq |((A - \lambda I)x, x)_H| \]

\[ \geq \text{Im}((A - \lambda I)x, x)_H \geq |\text{Im}(\lambda)| \| x \|_H^2 \quad \forall x \in D(A). \]

Here we used that \((Ax, x)_H\) is real for all \( x \) in \( D(A) \). Then \((\lambda I - A)^{-1}\) exists for \( \text{Im}(\lambda) \neq 0 \). In 
addition, \( R(\lambda I - A) \) must be dense in \( H \) if \( \text{Im}(\lambda) \neq 0 \). For suppose not. Then there exists a 
\( y \neq 0 \) which is orthogonal to \( R(\lambda I - A) \). But in that case we would have

\[ ((A - \lambda I)x, y)_H = 0 \quad \forall x \in D(A) \]

and since \( A \) is self adjoint, this is the same as
\[(x, (A - \bar{\lambda}I)y)_H = 0 \quad \forall x \in D(A).\]

But \(D(A)\) is dense in \(H\) so this implies \((A - \bar{\lambda}I)y = 0\), which is to say \(Ay = \bar{\lambda}y\). Since \((Ay, y)_H\) is real for all \(y\) there can be no such \(y\).

If \(A\) is a closed, densely defined linear operator on \(H\), then the resolvent set is an open subset of the complex plane. In each component of the resolvent set, \((\lambda I - A)^{-1}\) is an analytic function of \(\lambda\) with values in \(L(H)\). This can be seen by noting that for \(\lambda_0\) in the resolvent set of \(A\), \((\lambda_0 I - A)^{-1}\) is a bounded linear operator on \(H\) with domain equal to \(R(\lambda I - A) = H\). Fix \(\lambda_0\) in the resolvent set and define

\[S(\lambda) = (\lambda_0 I - A)^{-1}\left\{I + \sum_{n=1}^{\infty} (\lambda_0 - \lambda)^n [(\lambda_0 I - A)^{-1}]^n\right\}.

This series converges in the norm of \(L(H)\) provided \(\lambda\) lies in the disc \(|(\lambda_0 - \lambda)| ||(\lambda_0 I - A)^{-1}|| < 1\). Then the series defines an analytic function of \(\lambda\) in this disc. Now it is an exercise to show that

\[
(\lambda I - A) S(\lambda) = [(\lambda - \lambda_0) I + (\lambda_0 I - A)] S(\lambda) = I
\]

\[
S(\lambda) (\lambda I - A) = I.
\]

Then \(S(\lambda) = (\lambda I - A)^{-1}\) and \(S(\lambda)\) is analytic in a disc.

Note that if both \(\lambda, \mu\) belong to the resolvent set for \(A\) then the following analogue of a partial fractions decomposition is valid.

\[
(\lambda I - A)^{-1} = (\lambda I - A)^{-1}(\mu I - A)(\mu I - A)^{-1}
\]

\[
= (\lambda I - A)^{-1}[(\mu - \lambda) I + (\lambda I - A)](\mu I - A)^{-1}
\]

\[
= (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1} + (\mu I - A)^{-1},
\]

and

\[
(\lambda I - A)^{-1} - (\mu I - A)^{-1} = (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1};
\]

i.e.,

\[
(\lambda I - A)^{-1}(\mu I - A)^{-1} = \frac{1}{\mu - \lambda} [(\lambda I - A)^{-1} - (\mu I - A)^{-1}].
\]