## Linear Operators

For reference purposes, we will collect a number of useful results regarding bounded and unbounded linear operators.

### Bounded Linear Operators

Suppose T is a bounded linear operator on a Hilbert space H. In this case we may suppose that the domain of T, $D(T)$, is all of H. For suppose it is not. Then let $D(T)^{CL}$ denote the closure of $D(T)$, and extend T to the closure by continuity. That is, if $x \in D(T)^{CL}$ then there exists a sequence $\{x_n\} \subset D(T)$ such that $x_n \to x \ in \ H$. Since T is continuous, $Tx_n$ converges to some limit $f \in H$ and we can define $Tx = f$. It is not difficult to show that the value $f$ depends on x but does not depend on the sequence $\{x_n\}$. If $D(T)^{CL}$ is not all of H, then it is a closed subspace of H and, as such, has an orthogonal complement in H. Then extend T by zero to this orthogonal complement to obtain a linear operator which is now defined on all of H. It is easy to show that this extended operator is still continuous and has the same norm as T.

Recall that the linear operator $T$ is said to be **well defined** on H if $Tv = f, \ and \ Tv \ = g \ \ iff \ \ f = g$. This is equivalent to the statement that there is no nonzero element f such that $T0 = f$. The linear operator $T$ is said to be **one to one** on H if $Tv = f, \ and \ Tu \ = f \ \ iff \ \ u = v$. This is equivalent to the statement that $Tu = 0 \ iff \ u = 0$, (only the zero element is mapped to zero).

### Adjoint of a Bounded Linear Operator

For T a bounded linear operator on Hilbert space H and a fixed v in H, let

$$F(u) = (Tu, v)_H \ \forall u \in H.$$

Clearly this defines a bounded linear functional F on H and it follows then from the Riesz theorem that there exists a unique element $z_F \in H$ such that

$$(Tu, v)_H = (u, z_F)_H \ \forall u \in H.$$

If the correspondence relating $z_F \ to \ v$ is denoted by $z_F = T^*v, \ then$

$$(Tu, v)_H = (u, T^*v)_H \ \forall u \in H.$$

Then it is not hard to show $T^*$ is well defined on H with values in H and, moreover, $T^*$ is linear and bounded with $\|T^*\| = \|T\|$. For example, if $T^*v = f, \ and \ T^*v \ = g$ then

$$(Tu, v)_H = (u, T^*v)_H = (u, f)_H \quad and \quad (Tu, v)_H = (u, T^*v)_H = (u, g)_H$$

hence

$$(u, f-g)_H = 0 \quad \forall u \in D(T) = H.$$

Then $f = g$ and $T^*$ is well defined. The operator $T^*$ is called the adjoint of T and we have seen it is a well defined and bounded linear operator given only that T is bounded. If, in addition, T is onto, then the adjoint is one to one. To see this suppose T is onto and $T^*v = 0$ for some v in H. Then for arbitrary f in H, there is some u in H such that $Tu = f$ and

$$(f, v)_H = (Tu, v)_H = (u, T^*v)_H = 0;$$

i.e. $v \perp H$ which implies v is zero and so $T^*$ is one to one. Conversely, suppose $T^*$ is one to one and $f \perp Rng\,T.$ That is,

$$0 = (Tu,f)_H = (u,T^*f)_H \qquad \forall u \in H$$

But this implies $T^*f \perp H,\ or\ T^*f = 0$. But we assumed $T^*$ is one to one and it follows that $f = 0$. Then $f \perp Rng\,T$ implies $f = 0$ which is to say, T is onto.

A bounded linear operator T such that $(Tu,v)_H = (u,Tv)_H$ for all u,v in H is said to be self adjoint. We are going to be interested in differential operators which are self adjoint but differential operators are typically not bounded from H into itself.

## Unbounded Linear Operators

Consider the operator

$$Tu(x) = u'(x), \qquad for\ u \in D(T) = C^1[0,1] \subset L^2[0,1] = H.$$

Then $D(T)$ is dense in H but it is not all of H. Note that $\{\phi_n(x)\} = \{\sin(n\pi x)\}$ is a sequence in $D(T)$ that is bounded in H but the sequence $\{T\phi_n(x)\} = \{(n\pi)\cos(n\pi x)\}$ is an unbounded sequence in H. Then T is densely defined and unbounded on H. Note that if we took H to be the Hilbert space $H^1[0,1]$ then T is defined on all of H but T does not map H into H. Note further that if we define

$$Su(x) = u'(x), \qquad for\ u \in D(S) = H^1[0,1] \subset L^2[0,1] = H,$$

then $D(T) \subset D(S) \subset H\ and\ Tu = Su$ for all u in $D(T)$. Then T is a restriction of S and S is an extension of T and both of these operators are densely defined in H. When dealing with differential operators on a Hilbert space of functions, we will generally try to choose the largest possible domain for the operator. However, as we shall see, this is not always easy to do.

**Adjoint of an Unbounded Linear Operator**

Suppose T is an unbounded but densely defined linear operator on H. Then we define $y \in D(T^*) \subset H\ with\ T^*y = v\ to\ mean$

$$(Tu,y)_H = (u,v)_H \qquad \forall u \in D(T).$$

$T^*$ is well defined because T is densely defined. To see this, suppose $T^*y = v\ and\ T^*y = w.$ Then

$$(Tu,y)_H = (u,v)_H = (u,w)_H \qquad \forall u \in D(T).$$

But in this case $(u,v-w)_H = 0 \quad \forall u \in D(T)$, and since $D(T)$ is dense in H, this implies $v = w$. It is easy to show that $D(T^*)$ is a nonempty subspace of H (i.e. $D(T^*)$ contains at least the zero element) and $T^*$ is a linear operator from $D(T^*)\ into\ H.$

For example, consider

$$Tu(x) = u'(x),$$
$$for \quad u \in D(T) = \{u \in C^1[0,1] : u(0) = u(1) = 0\} \subset L^2[0,1] = H.$$

Then

$$(Tu,v)_H = \int_0^1 u'(x)\,v(x)dx = u(x)\,v(x)|_{x=0}^{x=1} - \int_0^1 u(x)\,v'(x)\,dx = (u,\,-v')_H$$

Since this string of equalities is valid for any v in $D(T) = C^1[0,1]$, it follows that $D(T^*)$ contains $D(T)$, and moreover, $T^*v = -v'(x)$. In fact, we can show that

$$D(T^*) = C^1[0,1] \quad \big(\text{with no boundary conditions}\big).$$

On the other hand, consider

$$Tu(x) = u'(x),$$
$$for \quad u \in D(T) = \{u, u' \in L^2[0,1] : u(0) = 0\}.$$

Then $y \in D(T^*) \subset H \; with \; T^*y = v \; means$

$$(Tu,y)_H = (u,v)_H \qquad \forall u \in D(T).$$

Let $\quad F(x) = \int_1^x v, \quad so\ that \quad F' = v \quad and\ F(1) = 0.$

Then

$$(u,v)_H = \int_0^1 u(x)\,F'(x)dx = Fu|_{x=0}^{x=1} - \int_0^1 u'(x)\,F(x)dx = -(Tu,F)_H$$

and

$$(Tu,y+F)_H = 0 \qquad \forall u \in D(T).$$

Now choose $\quad u_0(x) = \int_0^x [y(s) + F(s)]\,ds,$

which implies

$$u_0 \in L^2[0,1], \quad u_0' = y + F \in L^2[0,1], \; and\ u_0(0) = 0;$$

i.e., $u_0 \in D(T)$. Moreover, then $\quad (Tu_0,y+F)_H = \|y+F\|_H^2 = 0,$

which leads to

$$y(x) = -F(x) = -\int_1^x v,$$

or

$$-y'(x) = v(x) = T^*y, \quad y,y' \in L^2[0,1], \quad y(1) = 0.$$

We can deduce from this that $T^*y = -y'(x),\; D(T^*) = \{y,y' \in L^2[0,1], \quad y(1) = 0\}$

A densely defined linear operator T on H is said to be symmetric if

$$(Tu,v)_H = (u,Tv)_H \qquad \forall u,v \in D(T).$$

In this case $D(T) \subset D(T^*)\ and\ T^*u = Tu \quad \forall u \in D(T)$. Then $T^*$ is an extension of T. If it is the case that T is symmetric and, in addition, $D(T) \subset D(T^*)$, then we say T is self-adjoint.

For any bounded, symmetric operator T, $D(T) = D(T^*) = H$ so the operator is then self adjoint; i.e., for bounded operators symmetry and self adjointness are equivalent. For unbounded operators, they are in general different. For example

$$T_1 u(x) = u''(x) \quad with \quad D(T_1) = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$$

can be easily shown to be self adjoint, in contrast to

$$T_2 u(x) = u''(x) \quad with \quad D(T_2) = \{u \in C^2[0,1] : u(0) = u(1) = 0, u'(0) = u'(1) = 0\}$$

which is symmetric but not selfadjoint. Similarly,

$$T_3 u(x) = i u'(x) \quad with \quad D(T_3) = \{u \in C^1[0,1] : u(0) = u(1) = 0\}$$

is symmetric but not selfadjoint while

$$T_4 u(x) = i u'(x) \quad with \quad D(T_4) = \{u \in C^1[0,1] : u(0) = 0\}$$

is neither symmetric nor selfadjoint

**Closed Linear Operators**

Recall that a linear operator T on H is said to be bounded if there exists a constant $C > 0$ such that $\|Tx\|_H \le C\|x\|_H$ for all x in H. T is said to be continuous if $x_n \to x \text{ in } H \text{ implies } Tx_n \to Tx \text{ in } H.$ For linear operators, these two notions are equivalent. A densely defined linear operator T is said to be closed if

$$\left\{ \begin{array}{c} x_n \in D(T) \\ x_n \to x \text{ in } H \\ Tx_n \to f \text{ in } H \end{array} \right\} \text{ implies } \left\{ \begin{array}{c} x \in D(T) \\ Tx = f \end{array} \right\}$$

Note that T continuous implies T closed but not conversely. Differential operators are not continuous but are closed, (if the domain is properly chosen). We have seen that if T is linear and densely defined, then the adjoint $T^*$ is well defined. In addition, $T^*$ is closed, whether or not T itself is closed. To see this, suppose $x_n \in D(T^*),\ x_n \to x \text{ in } H \text{ and } T^*x_n \to f \text{ in } H.$ Then for any $z \in D(T)$,

$$(Tz, x_n)_H = (z, T^*x_n)_H, \qquad (Tz, x_n)_H \to (Tz, x)_H, \quad and \quad (z, T^*x_n)_H \to (z, f)_H,$$

which is to say, $(Tz, x)_H = (z, f)_H \ \forall z \in D(T)$. But this implies $x \in D(T^*) \text{ and } T^*x = f,$ so $T^*$ is closed. Note that it was not necessary to suppose that T was closed.

We have seen in previous examples that it may be possible to specify the domain of a differential operator in several different ways. For example, $Tu(x) = u'(x)$ could have any of the following subspaces of $L^2[0,1]$ as its domain,

$$D_1 = C^\infty[0,1] \subset D_2 = C^1[0,1] \subset D_3 = H^1[0,1].$$

We would like to choose the domain as large as possible, and moreover to have T on this domain be closed. To see how this might be done, recall that if T is linear and densely defined then the adjoint $T^*$ is linear and well defined. Now if $T^*$ is also densely defined, then the adjoint of $T^*$, denoted $T^{**}$, is also well defined and it is closed since it is an adjoint. More precisely, $z \in D(T^{**})$ *with* $T^{**}z = w$ *means*

$$(z, T^*y)_H = (w, y)_H = (T^{**}z, y)_H \qquad \forall y \in D(T^*).$$

But, for any $y \in D(T^*)$, $(z, T^*y)_H = (Tz, y)_H \quad \forall z \in D(T)$. This implies that $D(T) \subset D(T^{**})$ *and* $T^{**}z = Tz \ \forall z \in D(T)$; i.e., $T^{**}$ is an extension of T, and it is closed. Thus if the densely defined operator T is not closed but its adjoint $T^*$ is densely defined, then T has a closed extension which is just the adjoint of $T^*$. The domain of this extension must contain the limit points of all sequences $\{u_n\}$ in $D(T)$ for which the sequence $\{Tu_n\}$ also converges. This is not the same thing as the closure of $D(T)$ as the following theorem shows.

**Theorem** (**Closed Graph Theorem**) A closed operator on a closed domain is necessarily bounded.

The theorem shows that if $D(T^{**})$ were equal to $D(T)^{CL}$ then the closed operator $T^{**}$ on the closed domain $D(T)^{CL}$ would be necessarily bounded. But this would contradict the fact that $T^{**}z = Tz \ \forall z \in D(T)$ and T is not bounded. Then, what we have instead is the following,

$$D(T) \subset D(T^{**}) \subset D(T)^{CL} = H,$$

where T is extended to the larger domain $D(T^{**})$ which is also dense in H but is not closed. In summary, we can say that if T is a densely defined linear operator on H then

a) if $T^*$ is also densely defined then $T^{**}$ is well defined
and $T^{**}$ is a closed extension of T

b) if T is closed then $T^*$ is densely defined and $T = T^{**}$

**Solvability of Linear Equations**

Let A denote a densely defined linear operator on H, and let

$$R(A) = \{y \in H : y = Ax \text{ for some } x \in D(A)\}$$
$$N(A) = \{x \in D(A) : Ax = 0\}.$$

Since A is densely defined, $A^*$ is well defined and we will let

$$R(A^*) = \{y \in H : y = A^*x \text{ for some } x \in D(A^*)\}$$

$$N(A^*) = \{x \in D(A^*) : A^*x = 0\}$$

Then we have the following facts relating these subspaces:

1. $R(A)^{\perp} = N(A^*)$

2. $(R(A)^{\perp})^{\perp} = R(A)^{CL} = N(A^*)^{\perp}$

To see that 1 is true, suppose first that $z \in R(A)^{\perp}$. Then $(Au,z)_H = 0 \ \forall u \in D(A)$. Then $z \in D(A^*)$, with $A^*z = 0$ since

$$0 = (Au,z)_H = (u,A^*z)_H \quad \forall u \in D(A).$$

This proves that $R(A)^{\perp} \subset N(A^*)$. now suppose $z \in N(A^*)$, $i.e.$, $(x,A^*z)_H = 0 \quad \forall x \in H$. In particular, if $x \in D(A)$ then

$$0 = (x,A^*z)_H = (Ax,z)_H \quad \forall x \in D(A).$$

But this implies $z \perp R(A)$ and we have proved that $N(A^*) \subset R(A)^{\perp}$. The two inclusions together imply 1. The proof of 2 is left as an exercise. Note that 2 implies that if A has closed range, then the equation $Au = f$ is solvable if and only if $f \perp N(A^*)$. In fact, we can show that

$$R(A) = R(A)^{CL} \quad \textit{if and only if} \quad R(A^*) = R(A^*)^{CL}$$

and if this is the case, then

$$R(A) = N(A^*)^{\perp} \qquad \textit{and} \qquad R(A^*) = N(A)^{\perp}$$

**Lemma 1** Suppose A is closed and densely defined, and for some $C > 0$,

$$\|Au\|_H \geq C\|u\|_H \qquad \forall u \in D(A).$$

Then A has closed range.

Proof- It is an exercise to show that the null space of any closed linear operator is a closed subspace of H. Then for A a closed linear operator, we have $H = N(A) \oplus N(A)^{\perp}$. Now let $\{v_n\} \subset D(A)$ be such that $Av_n = f_n \to f \text{ in } H$; i.e., f is a limit point of $R(A)$.
Now $v_n$ can be written uniquely in the form

$$v_n = u_n + w_n \qquad u_n \in N(A)^{\perp} \qquad \textit{and} \quad w_n \in N(A).$$

Then for each n, $\qquad Av_n = Au_n$

and

$$\|f_n - f_m\|_H = \|Au_n - Au_m\|_H \geq C\|u_n - u_m\|_H.$$

But then $\{u_n\}$ is a Cauchy sequence in H with limit point $u \in H$, and since A is closed we have

$$\left\{ \begin{array}{c} Au_n \to f \\ u_n \to u \end{array} \right\} \quad \text{implies} \quad \left\{ \begin{array}{c} u \in D(A) \\ Au = f \end{array} \right\}.$$

Then $f \in R(A)$ and it follows that A has closed range.■

**Corollary**- Suppose A is closed and densely defined, and for some $C > 0$,

$$(Au, u)_H \geq C\|u\|_H^2 \qquad \forall u \in D(A).$$

Then A has closed range.

An operator A with the property of the corollary is said to be accretive (and -A is said to be dissipative).

**Lemma 2**- Suppose A is densely defined and self adjoint, and for some $C > 0$,

$$(Au, u)_H \geq C\|u\|_H^2 \qquad \forall u \in D(A) \cap N(A)^{\perp}.$$

Then precisely one of the following alternatives must hold:

a) $Au = f$ has a unique solution for every $f \in H$

b) $Au = f$ has no solution unless $f \perp N(A^*)$.
   If $f \perp N(A^*)$ then solutions exist but are not unique.

It is evident why this is true. The hypotheses imply that $R(A)$ is closed, hence $R(A) = N(A^*)^{\perp}$. Then $R(A)$ is either all of H or else the orthogonal complement of $R(A)$ is just $N(A^*)$. In addition, since A is self adjoint, $N(A) = N(A^*)$ so the following statements are all equivalent, $N(A^*) = \{0\}$, $R(A) = H$, $N(A) = \{0\}$.

**Invertibility of A**

For an unbounded operator A, we define the inverse operator, $A^{-1}$, by

$$f \in D(A^{-1}) \text{ with } A^{-1}f = u \qquad \text{if and only if} \quad u \in D(A) \text{ with } Au = f.$$

Then for $u, v \in D(A)$, $\left\{\begin{array}{c} A^{-1}f = u \\ A^{-1}f = v \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} Au = f \\ Av = f \end{array}\right\} \Rightarrow A(u-v) = 0$

from which it is clear that $A^{-1}$ is well defined if and only if A is one to one.

In particular, if A is densely defined and if there exists $C > 0$ such that

$$(Au, u)_H \geq C\|u\|_H^2 \qquad \forall u \in D(A).$$

then $A^{-1}$ exists. In addition, in this case $A^{-1}$ is necessarily bounded. To see this note that if $A^{-1}$ were not bounded then we could find a sequence $\{u_n\} \subset D(A)$ with $\|u_n\|_H = 1$ *and* $\|Au_n\|_H \to 0$. This would contradict the accretiveness assumption.

If A is one to one and closed, then $A^{-1}$ is well defined and is also closed. To see this, suppose $\{f_n\} \subset R(A)$ is such that $f_n \to f,$ *and* $A^{-1}f_n \to u$. To show $A^{-1}$ is closed we have to show $f \in R(A)$ *and* $A^{-1}f = u$. Then let $u_n = A^{-1}f_n$ and note that since A is closed, we have that

$$\left\{\begin{array}{c} u_n \to u, \\ Au_n \to f \end{array}\right\} \quad \textit{implies} \quad \left\{\begin{array}{c} u \in D(A) \\ Au = f \end{array}\right\}$$

But this means $f \in R(A)$ *and* $A^{-1}f = u$ and we are done.

If A is one to one and closed, then $R(A)$ is closed if and only if $A^{-1}$ is bounded. To see why this is true suppose first that $A^{-1}$ is bounded and let $\{f_n\}$ a sequence in $R(A)$ with $f_n \to f$. Since $A^{-1}$ is bounded, $A^{-1}f_n = u_n$ is a Cauchy sequence in H with limit $u$. Then, since A is closed,

$$\left\{\begin{array}{c} u_n \to u, \\ Au_n = f_n \to f \end{array}\right\} \quad \textit{implies} \quad \left\{\begin{array}{c} u \in D(A) \\ Au = f \end{array}\right\}$$

Then $f \in R(A)$ and it follows that $R(A)$ is closed. Conversely, if $R(A)$ is closed then it follows from the closed graph theorem that $A^{-1}$ is bounded (since $A^{-1}$ is known to be closed).

If A is closed and densely defined on H then

$$\begin{array}{lll} & R(A) = H \quad \textit{iff} \quad (A^*)^{-1} \textit{ is bounded} \\ \textit{and} & R(A^*) = H \quad \textit{iff} \quad (A)^{-1} \textit{ is bounded.} \end{array}$$

To see this, suppose first $R(A) = H$. Since $(Ax, y)_H = (x, A^*y)_H \quad \forall x \in D(A)$, we conclude that $A^*y = 0$ *iff* $y = 0$. Then $(A^*)^{-1}$ is well defined, and since $A^*$ is closed, $(A^*)^{-1}$ is closed. But $R(A) = H$ is closed so $R(A^*)$ is also closed, and then, by the closed graph theorem, $(A^*)^{-1}$ is bounded. Conversely, suppose $(A^*)^{-1}$ is bounded.Then $N(A^*) = \{0\}$ and it follows that $R(A^*)$ is closed. But then $R(A)$ is closed and equal to $N(A^*)^{\perp} = H$.

**Spectral Theory for Closed Operators**

Let A be closed and densely defined on H. Then for every complex number $\lambda$, $\lambda I - A$ is also closed and has the same domain as A. We define the following terms

$\lambda$ belongs to the **resolvent set** for A if $(\lambda I - A)^{-1}$ exists and is bounded with $R(\lambda I - A) = H$

$\lambda$ belongs to the **point spectrum** for A if $(\lambda I - A)^{-1}$ fails to exist;
$i.e.,\ \ \exists u \neq 0$ such that $(\lambda I - A)u = H = 0$. Then $\lambda$ is an eigenvalue for A and $x \in N(\lambda I - A)$ is an eigenvector for A

$\lambda$ belongs to the **continuous spectrum** for A if $(\lambda I - A)^{-1}$ exists but is not bounded with

$$R(\lambda I - A) \neq H \quad but \quad R(\lambda I - A)^{CL} = H$$

$\lambda$ belongs to the **residual or compression spectrum** for A if $R(\lambda I - A)^{CL} \neq H$

We collect here several facts about the resolvent set and spectrum of A.

If A is closed, densely defined and symmetric then essentially the same proofs that work in linear algebra for symmetric matrices can be used to show that

$(Au, u)_H$ is real for all u in $D(A)$
all eigenvalues of A are real
eigenvectors corresponding to distinct eigenvalues are orthogonal;
i.e., $\lambda \neq \mu \ implies \ N(\lambda I - A) \perp N(\mu I - A)$.

If, in addition, A is self adjoint on H, then the resolvent set of A contains the complement of the real axis. Moreover, for $\mathrm{Im}(\lambda) \neq 0,\ (\lambda I - A)^{-1}$ is bounded and satisfies

$$||(\lambda I - A)^{-1}||_{L(H)} \leq \frac{1}{|\mathrm{Im}(\lambda)|}$$

and

$$Im((A - \lambda I)x, x)_H = \mathrm{Im}(\lambda)\ \|x\|_H^2 \qquad \forall x \in D(A).$$

This can be seen by writing

$$\|(A - \lambda I)x\|_H \|x\|_H \geq |((A - \lambda I)x, x)_H|$$

$$\geq Im((A - \lambda I)x, x)_H \geq |\mathrm{Im}(\lambda)|\ \|x\|_H^2 \qquad \forall x \in D(A).$$

Here we used that $(Ax, x)_H$ is real for all x in $D(A)$. Then $(\lambda I - A)^{-1}$ exists for $\mathrm{Im}(\lambda) \neq 0$. In addition, $R(\lambda I - A)$ must be dense in H if $\mathrm{Im}(\lambda) \neq 0$. For suppose not. Then there exists a $y \neq 0$ which is orthogonal to $R(\lambda I - A)$. But in that case we would have

$$((A - \lambda I)x, y)_H = 0 \qquad \forall x \in D(A)$$

and since A is self adjoint, this is the same as

$$(x,(A-\bar{\lambda}I)y)_H = 0 \qquad \forall x \in D(A).$$

But $D(A)$ is dense in H so this implies $(A-\bar{\lambda}I)y = 0,$ which is to say $Ay = \bar{\lambda}y.$ Since $(Ay,y)_H$ is real for all y there can be no such y.

If A is a closed, densely defined linear operator on H, then the resolvent set is an open subset of the complex plane. In each component of the resolvent set, $(\lambda I - A)^{-1}$ is an analytic function of $\lambda$ with values in $L(H)$. This can be seen by noting that for $\lambda_0$ in the resolvent set of A, $(\lambda_0 I - A)^{-1}$ is a bounded linear operator on H with domain equal to $R(\lambda I - A) = H.$ Fix $\lambda_0$ in the resolvent set and define

$$S(\lambda) = (\lambda_0 I - A)^{-1}\left\{ I + \sum_{n=1}^{\infty}(\lambda_0 - \lambda)^n \left[ (\lambda_0 I - A)^{-1} \right]^n \right\}.$$

This series converges in the norm of $L(H)$ provided $\lambda$ lies in the disc $|(\lambda_0 - \lambda)|$ $\left|\left|(\lambda_0 I - A)^{-1}\right|\right| < 1.$ Then the series defines an analytic function of $\lambda$ in this disc. Now it is an exercise to show that

$$(\lambda I - A)\,S(\lambda) = [(\lambda - \lambda_0)I + (\lambda_0 I - A)]\,S(\lambda) = I$$
$$S(\lambda)\,(\lambda I - A) = I.$$

Then $S(\lambda) = (\lambda I - A)^{-1}$ and $S(\lambda)$ is analytic in a disc.

Note that if both $\lambda, \mu$ belong to the resolvent set for A then the following analogue of a partial fractions decomposition is valid.

$$(\lambda I - A)^{-1} = (\lambda I - A)^{-1}(\mu I - A)(\mu I - A)^{-1}$$

$$= (\lambda I - A)^{-1}[(\mu - \lambda)I + (\lambda I - A)](\mu I - A)^{-1}$$

$$= (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1} + (\mu I - A)^{-1},$$

and

$$(\lambda I - A)^{-1} - (\mu I - A)^{-1} = (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1};$$

i.e.,

$$(\lambda I - A)^{-1}(\mu I - A)^{-1} = \frac{1}{\mu - \lambda}\left[ (\lambda I - A)^{-1} - (\mu I - A)^{-1} \right].$$

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