Equations of Fluid Flow

M 646

Conservation Equations

We suppose that a region Ω in R^3 is filled with fluid and that U denotes an arbitrary subregion within Ω. The state of the fluid system can be described in terms of the following state variables,

$\rho = \rho(\bar{x}, t) =$density field (scalar)
$\bar{u} = \bar{u}(\bar{x}, t) =$velocity field (vector)

Conservation of Mass

For an arbitrary subregion U in Ω, we have

$$\frac{d}{dt}\int_U \rho\, dx = \int_U \partial_t \rho\, dx = \text{the rate of change of mass in } U$$

$$\int_{\partial U} \rho\, \bar{u} \cdot \bar{n}\, dS = \text{net outflow rate from } U, \quad \bar{n} = \textit{unit outward normal to } \partial U$$

The divergence theorem implies

$$\int_{\partial U} \rho\, \bar{u} \cdot \bar{n}\, dS = \int_U div\,(\rho \bar{u})\, dx,$$

and thus

$$\int_U [\partial_t \rho + div\,(\rho \bar{u})]\, dx = 0, \qquad \textit{for all } U \subset \Omega,$$

or

$$\partial_t \rho + div\,(\rho \bar{u}) = 0 \qquad \text{at each point in } \Omega.$$

These last two equations are the integral and differential form of the so called equation of continuity in fluid flow. If the density and velocity fields are not sufficiently smooth, the differential equation must be interpreted in a weak sense.

Conservation of Momentum

If we define

$$\int_U \rho\, \bar{u}\, dx = \text{the momentum of the fluid in U}$$

then $$\frac{d}{dt}\int_U \rho\, \bar{u}\, dx = \text{sum of the forces acting on } U$$

$$= \text{external body forces (gravity)} + \text{internal forces (friction)}$$

$$= \int_U \rho\, \bar{F}\, dx + \int_{\partial U} [\sigma]\, \bar{n}\, dS,$$

where

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{bmatrix} = \text{stress tensor}$$

and

$$\int_{\partial U} [\sigma]\,\bar{n}\,dS = \int_U div\,[\sigma]\,dx = \int_U \begin{bmatrix} \partial_x\sigma_{11} + \partial_y\sigma_{12} + \partial_z\sigma_{13} \\ \vdots \\ \partial_x\sigma_{31} + \partial_y\sigma_{32} + \partial_z\sigma_{33} \end{bmatrix} dx.$$

This leads (eventually) to the momentum equation for the fluid flow,

$$\rho\,\frac{d}{dt}\bar{u} - \rho\bar{F} - div\,[\sigma] = \bar{0} \qquad in\ \Omega.$$

Here $\quad \dfrac{d}{dt}\bar{u} = \partial_t\bar{u} + \partial_x\bar{u}\,\dfrac{dx}{dt} + \partial_y\bar{u}\,\dfrac{dy}{dt} + \partial_z\bar{u}\,\dfrac{dz}{dt} = \partial_t\bar{u} + \bar{u}\bullet\nabla\bar{u},$

so $\quad \rho\,(\partial_t\bar{u} + \bar{u}\bullet\nabla\bar{u}) - \rho\bar{F} - div\,[\sigma] = \bar{0} \qquad in\ \Omega.$

Every fluid will be assumed to satisfy the two conservation equations. Additional equations may be required to completely determine the flow.

Special Flows

We consider now a number of special cases of the flow equations.

1. **Incompressible Flow** $\qquad \rho = \rho_0$ **(constant)**

Then the continuity equation becomes $\qquad div\,\bar{u} = 0 \qquad in\ \Omega$

If we suppose, in addition, that $\qquad curl\ \bar{u} = \bar{0} \qquad in\ \Omega$

then the flow is said to be incompressible and irrotational (it can be argued that the flow is then energy conserving) and it is a well known result from vector calculus that $curl\ \bar{u} = \bar{0}$ if and only if $\bar{u} = grad\ \phi$ for some smooth $\phi = \phi(x)$. In this case, the continuity equation becomes $div\,\bar{u} = div\left(grad\ \phi\right) = \nabla^2\phi = 0 \quad in\ \Omega$. Such a flow is called a *potential flow*. and it is no longer a system of equations but a single pde (Laplace's equation) whose solution, subject to boundary constraints, leads to complete determination of the velocity field via $\bar{u} = grad\ \phi$.

2. **Inviscid** (**Frictionless**) **Flow** $\qquad [\sigma] = -p(x,t)\,[I] \qquad p = pressure$

The only internal forces are pressure forces. Then we can have

(**a**) **Inviscid**, **Compressible**

$$\left.\begin{array}{c} \partial_t\rho + div\,(\rho\bar{u}) = 0\ \ in\ \Omega \\ \rho\,(\partial_t\bar{u} + \bar{u}\bullet\nabla\bar{u}) + \nabla p = \rho\bar{F} \qquad in\ \Omega. \end{array}\right\}(4\ eqs,\ 5\ unknowns)$$

$$p = f(\rho) \quad equation\ of\ state \qquad \text{(fifth eq)}$$

These are the equations of gas dynamics.

(**b**) **Inviscid**, **Incompressible** (**Euler's equations**)

$$\rho = \rho_0$$
$$div\,\bar{u} = 0 \qquad in\ \Omega$$
$$\partial_t\bar{u} + \bar{u}\bullet\nabla\bar{u} + \frac{1}{\rho_0}\nabla p = \bar{F} \qquad in\ \Omega.$$

This is referred to as *ideal flow*.

3. **Viscous Flow** $[\sigma] = -p(x,t)\,[I] + \psi([T])$

$$[T] = [grad\,\bar{u}] = \begin{bmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x w & \partial_y w & \partial_z w \end{bmatrix}$$

Empirical evidence suggests that ψ should be linear in T and it should be invariant under translation and rotation of coordinates. This leads to

$$[\sigma] = -p(x,t)\,[I] + \mu'\,div\,\bar{u}\,[I] + \underbrace{\mu\,([T] + [T^*])}_{symmetric\ part\ of\ T}$$

μ, μ' =material constants.

Then we can have the following special cases of the viscous flow case:

(a) **Viscous**, **compressible flow**

$$\partial_t\rho + div\,(\rho\bar{u}) = 0\ \ in\ \Omega$$
$$\rho\,(\partial_t\bar{u} + \bar{u}\bullet\nabla\bar{u}) + \nabla p = \rho\bar{F} + \mu\nabla^2\bar{u} + (\mu + \mu')\,grad(div\,\bar{u}) \qquad in\ \Omega$$
$$p = f(\rho),$$

(b) **Viscous**, **incompressible flow** (**Navier**-**Stokes flow**)

$$\rho = \rho_0$$
$$div\,\bar{u} = 0 \qquad in\ \Omega$$
$$\partial_t\bar{u} + \bar{u}\bullet\nabla\bar{u} + \frac{1}{\rho_0}\nabla p = \bar{F} + \nu\nabla^2\bar{u} \qquad in\ \Omega$$

(c) **Steady-state**, **viscous**, **incompressible flow**

$$\rho = \rho_0$$
$$div\,\bar{u} = 0 \qquad in\ \Omega$$
$$-\nu\nabla^2\bar{u} + \bar{u}\bullet\nabla\bar{u} + \frac{1}{\rho_0}\nabla p = \bar{F} \qquad in\ \Omega$$

We view this solution as the limit to which the solutions to the N-S equations tend as t tends to infinity.

(c) **Linearized**, **viscous**, **incompressible flow**

$$\begin{aligned} \rho &= \rho_0 \\ div\,\bar{u} &= 0 && in\ \Omega \\ \partial_t \bar{u} - \nu \nabla^2 \bar{u} + \frac{1}{\rho_0}\nabla p &= \bar{F} && in\ \Omega \end{aligned}$$

We assume here that $\|\bar{u} \cdot \nabla \bar{u}\| \approx 0.$

(d) **Steady-state**, **linearized**, **viscous**, **incompressible flow**

$$\begin{aligned} \rho &= \rho_0 \\ div\,\bar{u} &= 0 && in\ \Omega \\ -\nu \nabla^2 \bar{u} + \frac{1}{\rho_0}\nabla p &= \bar{F} && in\ \Omega. \end{aligned}$$

This is known as *Stokes flow*.

(e) **Acoustic Waves**

We suppose

$$\begin{aligned} \rho &= \rho_0(1 + s(\bar{x}, t)) \qquad s << 1 \\ p &= F(\rho) = F(\rho_0) + F'(\rho_0)(\rho - \rho_0) \\ &= p_0 + \rho_0 F'(\rho_0)\, s(\bar{x}, t) \end{aligned}$$

and $\|\bar{u} \cdot \nabla \bar{u}\| \approx 0.$

Then $$\begin{aligned} &\partial_t(\rho_0(1 + s(\bar{x}, t))) + div(\rho_0 \bar{u}) = 0 \\ &\partial_t \bar{u} + F'(\rho_0)\nabla s(\bar{x}, t) = \bar{0}, \end{aligned}$$

and by combining $\partial_t\left(1st\ eqn\right)$ with $div\left(2nd\ eqn\right),$ we find that

$$\partial_{tt} s(\bar{x}, t) = F'(\rho_0)\, \nabla^2 s(\bar{x}, t);$$

i.e., $s(\bar{x}, t)$ satisfies the *wave equation* with $c^2 = F'(\rho_0)$.