

Examples Using Energy Estimates

We will use energy estimates for solutions to the N-S equations to derive existence results for solutions having various types of regularity. Of course the various types of regularity for the solution are determined by the regularity assumptions on the data in the problem and it will be our aim to show precisely how the energy estimates are modified to correspond to the data regularity in such a way as to lead to the various kinds of regularity for the solution.

Recall first, that in the basic existence proof we derived the following estimate for the nonlinear term

$$|b(\vec{u}, \vec{u}, \vec{v})| \leq C \|\vec{u}\|_V^2 \|\vec{v}\|_V \quad \text{for } \vec{u}, \vec{v} \in H_0^1(U)^n$$

and this eventually led to the result that $B(\vec{u}(t)) : L^2[0, T : V] \rightarrow L^p[0, T : V']$ for $p \leq 4/n$. Then the assumptions, $\vec{f}(t) \in L^2[0, T : V']$, $\vec{u}_0 \in H$, on the data lead to a weak solution $\vec{u}(t) \in L^2[0, T : V] \cap L^\infty[0, T : H]$, with $\vec{u}'(t) \in L^p[0, T : V']$ for $p \leq 4/n$.

Example 1 Here, we assume that $n = 2$, and

$$\begin{aligned} & \vec{f}(t) \text{ and } \vec{f}'(t) \in L^2[0, T : V'], \quad \vec{f}(0) \in H \\ \text{and} \quad & \vec{u}_0 \in H^2(U)^n \cap V. \end{aligned}$$

Then for $\{\vec{w}_j\}$ a family of functions in K (the divergence free test functions in U) that spans a dense subset of V , let

$$\vec{u}_N(t) = \sum_{j=1}^N C_{j,N}(t) \vec{w}_j$$

satisfy

$$(\vec{u}'_N(t), \vec{w}_k)_H + \nu [[\vec{u}_N(t), \vec{w}_k]] + b(\vec{u}_N(t), \vec{u}_N(t), \vec{w}_k) = (\vec{f}(t), \vec{w}_k)_H, \quad (1)$$

for $1 \leq k \leq N$.

Since $\vec{u}_0 \in H^2(U)^n \cap V$, we can also let $\vec{u}_N(0) = P_N[\vec{u}_0]$, where P_N denotes the orthogonal projection in $H^2(U)^n \cap V$ onto the subspace $V_N = \text{span}[\vec{w}_1, \dots, \vec{w}_N]$. Then clearly,

$$\|\vec{u}_N(0)\|_{H^2} = \|P_N[\vec{u}_0]\|_{H^2} \leq \|\vec{u}_0\|_{H^2} \text{ and } \|P_N[\vec{u}_0] - \vec{u}_0\|_{H^2} \rightarrow 0 \text{ as } N \rightarrow \infty$$

The initial value problem for (1) has a unique solution for each N and by multiplying (1) by $C'_{j,N}(t)$ and forming linear combinations, we can infer that

$$(\vec{u}'_N(t), \vec{u}'_N(t))_H + \nu [[\vec{u}_N(t), \vec{u}'_N(t)]] + b(\vec{u}_N(t), \vec{u}_N(t), \vec{u}'_N(t)) = (\vec{f}(t), \vec{u}'_N(t))_H,$$

At $t=0$ in particular, we have

$$\begin{aligned}\|\vec{u}'_N(0)\|_H^2 &= \left(\vec{f}(0), \vec{u}'_N(0)\right)_H - v[[\vec{u}_N(0), \vec{u}'_N(0)]] - b(\vec{u}_N(0), \vec{u}_N(0), \vec{u}'_N(0)) \\ &= \left(\vec{f}(0), \vec{u}'_N(0)\right)_H - v(A\vec{u}_N(0), \vec{u}'_N(0))_H - b(\vec{u}_N(0), \vec{u}_N(0), \vec{u}'_N(0))\end{aligned}$$

and, since we assumed that $\vec{f}(0) \in H$,

$$\|\vec{u}'_N(0)\|_H^2 \leq \|\vec{f}(0)\|_H \|\vec{u}'_N(0)\|_H + v \|A\vec{u}_N(0)\|_H \|\vec{u}'_N(0)\|_H + \|B\vec{u}_N(0)\|_H \|\vec{u}'_N(0)\|_H$$

i.e.,
$$\|\vec{u}'_N(0)\|_H \leq \|\vec{f}(0)\|_H + v \|A\vec{u}_N(0)\|_H + \|B\vec{u}_N(0)\|_H$$

Now observe that,

(i) $A\vec{u}_N(0) = -\nabla^2 \vec{u}_N(0)$ hence

$$\|A\vec{u}_N(0)\|_H = \|\nabla^2 \vec{u}_N(0)\|_H \leq C \|\vec{u}_N(0)\|_{H^2} \leq C \|\vec{u}_0\|_{H^2}$$

(ii) $|b(\vec{u}, \vec{u}, \vec{v})| \leq C \|\vec{u}\|_4 \|\nabla \vec{u}\|_4 \|\vec{v}\|_2$ (by the generalized Holder inequality)

The Sobolev inequality for $n=2$ implies,

$$\|\vec{u}\|_4 \leq \|\vec{u}\|_2^{1/2} \|\nabla \vec{u}\|_2^{1/2} \quad \text{and} \quad \|\nabla \vec{u}\|_4 \leq \|\nabla \vec{u}\|_2^{1/2} \|\nabla^2 \vec{u}\|_2^{1/2}$$

Then the Cauchy inequality gives,

$$\|\vec{u}\|_2^{1/2} \|\nabla \vec{u}\|_2^{1/2} \leq \|\vec{u}\|_V, \quad \|\nabla \vec{u}\|_2^{1/2} \|\nabla^2 \vec{u}\|_2^{1/2} \leq \|\vec{u}\|_{H^2}$$

and, finally, $|b(\vec{u}, \vec{u}, \vec{v})| \leq C \|\vec{u}\|_V \|\vec{u}\|_{H^2} \|\vec{v}\|_H$ for all $\vec{u} \in H^2(U)^n \cap V$ and all $\vec{v} \in H$. It follows that for each $\vec{u} \in H^2(U)^n \cap V$ there is a unique $B(\vec{u}) \in H$ such that $|b(\vec{u}, \vec{u}, \vec{v})| = |(B(\vec{u}), \vec{v})_H| \quad \forall \vec{v} \in H$.

Then

$$\begin{aligned}\|B\vec{u}_N(0)\|_H &\leq C \|\vec{u}_N(0)\|_V \|\vec{u}_N(0)\|_{H^2} \\ &\leq C \|\vec{u}_N(0)\|_{H^2}^2 \leq C \|\vec{u}_0\|_{H^2}\end{aligned}$$

These results lead to,

$$\|\vec{u}'_N(0)\|_H \leq \|\vec{f}(0)\|_H + vC \|\vec{u}_0\|_{H^2} + C \|\vec{u}_0\|_{H^2} = M_0(\vec{f}(0), \vec{u}_0)$$

i.e, $\{\vec{u}'_N(0)\}$ is a bounded sequence in H.

By differentiating the approximate equation with respect to t , and then forming linear combinations of the results, we get

$$\begin{aligned} (\vec{u}''_N(t), \vec{u}'_N(t))_H + \nu [[\vec{u}'_N(t), \vec{u}'_N(t)]] + b(\vec{u}'_N(t), \vec{u}_N(t), \vec{u}'_N(t)) \\ + b(\vec{u}_N(t), \vec{u}'_N(t), \vec{u}'_N(t)) = (\vec{f}'(t), \vec{u}'_N(t))_H \end{aligned}$$

Note that since we have assumed $\vec{f}'(t) \in L^2[0, T : V']$, then

$$(\vec{f}'(t), \vec{u}'_N(t))_H = \langle \vec{f}'(t), \vec{u}'_N(t) \rangle_{V' \times V}.$$

In addition, $(\vec{u}''_N(t), \vec{u}'_N(t))_H = \frac{1}{2} \frac{d}{dt} \|\vec{u}'_N(t)\|_H^2$

and since $|b(\vec{u}, \vec{v}, \vec{w})| \leq C(\|\vec{u}\|_H \|\vec{u}\|_V)^{1/2} \|\vec{v}\|_V (\|\vec{w}\|_H \|\vec{w}\|_V)^{1/2}$
for $n=2$,

$$\frac{d}{dt} \|\vec{u}'_N(t)\|_H^2 + 2\nu \|\vec{u}'_N(t)\|_V^2 \leq 2 \|\vec{f}'(t)\|_{V'} \|\vec{u}'_N(t)\|_V + 2C \|\vec{u}'_N(t)\|_H \|\vec{u}'_N(t)\|_V \|\vec{u}_N(t)\|_V$$

Now $2C \|\vec{u}'_N(t)\|_H \|\vec{u}'_N(t)\|_V \|\vec{u}_N(t)\|_V \leq \nu \|\vec{u}'_N(t)\|_V^2 + C(\nu) \|\vec{u}'_N(t)\|_H^2 \|\vec{u}_N(t)\|_V^2$

and $2 \|\vec{f}'(t)\|_{V'} \|\vec{u}'_N(t)\|_V \leq \frac{1}{2} \nu \|\vec{u}'_N(t)\|_V^2 + \nu/2 \|\vec{f}'(t)\|_{V'}^2$

therefore,

$$\frac{d}{dt} \|\vec{u}'_N(t)\|_H^2 + \frac{1}{2} \nu \|\vec{u}'_N(t)\|_V^2 \leq C \left[\|\vec{f}'(t)\|_{V'}^2 + \|\vec{u}'_N(t)\|_H^2 \|\vec{u}_N(t)\|_V^2 \right] \quad (2)$$

The approximate equation has been manipulated into the form of an energy estimate from which the basic a-priori estimates can be derived.

Note first that (2) implies

$$\frac{d}{dt} \|\vec{u}'_N(t)\|_H^2 - a_N(t) \|\vec{u}'_N(t)\|_H^2 \leq C \|\vec{f}'(t)\|_{V'}^2$$

where

$$a_N(t) = C \|\vec{u}_N(t)\|_V^2.$$

Since we showed in the course of proving the basic existence theorem that (under weaker assumptions on the data) $\|\vec{u}_N\|_{L^2[0, T; V]}$ is bounded, uniformly in N,

it follows that $a_N(t) \in L^1[0, T]$. Then the Gronwall inequality implies

$$\|\vec{u}'_N(t)\|_H^2 \leq \left[C \|\vec{f}'(t)\|_{V'}^2 + \|\vec{u}'_N(0)\|_H^2 \right] e^{\int_0^t a_N(s) ds} \leq C \left[\|\vec{f}'(t)\|_{V'}^2 + M_0 \right]$$

Since this holds for all t , $0 \leq t \leq T$, we have proved

$$\|\vec{u}'_N\|_{L^\infty[0, T; H]} \leq M_1$$

i.e., $\{\vec{u}'_N(t)\}$ is a bounded sequence in $L^\infty[0, T : H]$

A second a-priori estimate is obtained by integrating (2) from 0 to T. Then

$$\frac{1}{2} \nu \int_0^T \|\vec{u}'_N(t)\|_V^2 dt \leq C \left[\int_0^T \|\vec{f}'(t)\|_{V'}^2 dt + \int_0^T \|\vec{u}'_N(t)\|_H^2 \|\vec{u}_N(t)\|_V^2 dt \right]$$

$$\|\vec{u}'_N\|_{L^2[0, T; V]}^2 \leq C \left[\|\vec{f}'\|_{L^2[0, T; V']}^2 + \|\vec{u}'_N\|_{L^\infty[0, T; H]}^2 \|\vec{u}_N\|_{L^2[0, T; V]}^2 \right] \leq M_2$$

i.e., $\{\vec{u}'_N(t)\}$ is a bounded sequence in $L^2[0, T : V]$

These a-priori estimates now imply the existence of a subsequence converging weakly in $L^2[0, T : V]$ and weak-* in $L^\infty[0, T : H]$ to a limit $v(t) \in L^2[0, T : V] \cap L^\infty[0, T : H]$. In addition, we can use distributional arguments to show that $v(t) = u'(t)$ where $u(t)$ denotes the limit in $L^2[0, T : V]$ of the sequence $\{\vec{u}_N(t)\}$.

Then these hypotheses on the data have led to the existence of a weak solution $\vec{u}(t)$ of the N-S equations with

$$\vec{u}(t) \in L^2[0, T : V] \cap C[0, T : H] \quad \text{and} \quad u'(t) \in L^2[0, T : V] \cap L^\infty[0, T : H]$$

Example 2 Here we assume $n=2,3$ and $U \subset \mathbb{R}^n$ has a smooth (i.e., C^2) boundary. We will suppose the data satisfies,

$$\vec{f}(t) \in L^2[0, T : H], \quad \text{and} \quad \vec{u}_0 \in V.$$

For $\vec{u} \in H^2(U)^n \cap V$

$$(i) \quad \|A\vec{u}\|_H^2 = \|\nabla^2 \vec{u}\|_H^2 \leq \sum_{|\alpha| \leq 2} \|D^\alpha \vec{u}\|_H^2 = \|\vec{u}\|_{H^2}^2$$

(ii) $\vec{f} = A\vec{u} \in L^2(U)^n$ but then the regularity theorem for the Stokes' problem implies

$$\vec{u} \in H^2(U)^n \quad \text{and} \quad \|\vec{u}\|_{H^2}^2 \leq C \|\vec{f}\|_H^2 = C \|A\vec{u}\|_H^2$$

Then it is clear that $\|\vec{u}\|_A = \|A\vec{u}\|_H$ defines a norm on $H^2(U)^n \cap V$ that is equivalent to the $H^2(U)^n$ - norm.

Before proceeding with the energy estimates we need an estimate for the nonlinear term from the N-S equation.

Lemma 1 For $\vec{u} \in H^2(U)^n \cap V$, $B(\vec{u}) \in H$ and

$$(i) \quad \|B(\vec{u})\|_H \leq C_1 \|\vec{u}\|_H^{1/2} \|A\vec{u}\|_H^{1/2} \|\vec{u}\|_V \quad \text{if } n = 2$$

$$(ii) \quad \|B(\vec{u})\|_H \leq C_2 \|\vec{u}\|_V^{3/2} \|A\vec{u}\|_H^{1/2} \quad \text{if } n = 3$$

Proof of lemma- When $n=2$, apply the generalized Holder inequality with $p=q=4$, and $r=2$ so that

$$\left| \int_U u_i \partial_i u_j v_j dx \right| \leq \|u_i\|_4 \|\partial_i u_j\|_4 \|v_j\|_2$$

Now the Sobolev inequality for $n=2$ can be used to obtain

$$\begin{aligned} \left| \int_U u_i \partial_i u_j v_j dx \right| &\leq C (\|u_i\|_2 \|\nabla u_i\|_2)^{1/2} (\|\partial_i u_j\|_2 \|\nabla \partial_i u_j\|_2)^{1/2} \|v_j\|_2 \\ &\leq C \|u_i\|_H^{1/2} (\|\partial_i u_j\|_2 \|\nabla u_i\|_2)^{1/2} (\|\nabla \partial_i u_j\|_2)^{1/2} \|v_j\|_2 \end{aligned}$$

and it follows that,

$$|b(\vec{u}, \vec{u}, \vec{v})| \leq C \|\vec{u}\|_H^{1/2} (\|\vec{u}\|_V^2)^{1/2} (\|\vec{u}\|_H^2)^{1/2} \|\vec{v}\|_H \leq C \|\vec{u}\|_H^{1/2} \|\vec{u}\|_V \|\vec{u}\|_A^{1/2} \|\vec{v}\|_H$$

This estimate implies that for each $\vec{u} \in H^2(U)^n \cap V$, $b(\vec{u}, \vec{u}, \bullet)$ defines a bounded linear functional $B(\vec{u})$ on H with $\|B(\vec{u})\|_H \leq C \|\vec{u}\|_H^{1/2} \|\vec{u}\|_V \|\vec{u}\|_A^{1/2}$.

When $n=3$, apply the generalized Holder inequality with four functions and $p=6$, $q=4$, $r=12$ and $s=2$, so that

$$\begin{aligned} \left| \int_U u_i \partial_i u_j v_j dx \right| &= \left| \int_U u_i (\partial_i u_j)^{1/2} (\partial_i u_j)^{1/2} v_j dx \right| \\ &\leq \|u_i\|_6 \|(\partial_i u_j)^{1/2}\|_4 \|(\partial_i u_j)^{1/2}\|_{12} \|v_j\|_2 \\ &\leq \|u_i\|_6 \|(\partial_i u_j)\|_2^{1/2} \|(\partial_i u_j)\|_6^{1/2} \|v_j\|_2 \end{aligned}$$

Now the Sobolev embedding theorem asserts that for $n=3$, $H^1(U) \subset L^6(U)$ and so

this last estimate implies

$$\left| \int_U u_i \partial_i u_j v_j dx \right| \leq C \|u_i\|_{H^1}^{3/2} \|(\partial_i u_j)\|_{H^1}^{1/2} \|v_j\|_2 \quad \forall \vec{u} \in H^2(U)^n \cap V,$$

hence

$$|b(\vec{u}, \vec{u}, \vec{v})| \leq C \|\vec{u}\|_V^{3/2} (\|\vec{u}\|_{H^2})^{1/2} \|\vec{v}\|_H$$

which leads to the result (ii). ■

In this example, we take the basis $\{\vec{w}_j\}$ to be the eigenfunctions of the Stokes' operator A.

Then

$$((\vec{w}_j, \vec{v})) = (A\vec{w}_j, \vec{v})_H = \lambda_j (\vec{w}_j, \vec{v})_H \quad \forall \vec{v} \in V$$

and

$$((\vec{w}_j, \vec{w}_k)) = (A\vec{w}_j, \vec{w}_k)_H = \begin{cases} \lambda_j & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

If
$$\vec{u}_N(t) = \sum_{j=1}^N C_{j,N}(t) \vec{w}_j$$

solves
$$(\vec{u}'_N(t), \vec{w}_j)_H + (vA\vec{u}_N(t) + B(\vec{u}_N(t)), \vec{w}_j)_H = (\vec{f}(t), \vec{w}_j)_H \quad 1 \leq j \leq N,$$

$$\vec{u}_N(0) = \sum_{j=1}^N ((\vec{u}_0, \vec{w}_j)) \vec{w}_j,$$

then clearly, $\|\vec{u}_N(0) - \vec{u}_0\|_V \rightarrow 0$, as $N \rightarrow \infty$ since we have assumed that $u_0 \in V$.

Multiplying the approximate equation by λ_j leads to

$$((\vec{u}'_N(t), \vec{w}_j)) + (vA\vec{u}_N(t) + B(\vec{u}_N(t)), A\vec{w}_j)_H = (\vec{f}(t), A\vec{w}_j)_H \quad 1 \leq j \leq N,$$

and, forming linear combinations of these equations, we get

$$((\vec{u}'_N(t), \vec{u}_N(t))) + (vA\vec{u}_N(t) + B(\vec{u}_N(t)), A\vec{u}_N(t))_H = (\vec{f}(t), A\vec{u}_N(t))_H \quad 1 \leq j \leq N.$$

It follows from lemma 1 and the hypothesis that $\vec{f}(t) \in L^2[0, T : H]$, that we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\vec{u}_N(t)\|_V^2 + \nu \|A\vec{u}_N(t)\|_H^2 &\leq \|\vec{f}(t)\|_H \|A\vec{u}_N(t)\|_H + C_1 \|\vec{u}_N(t)\|_H^{1/2} \|A\vec{u}_N(t)\|_H^{3/2} \|\vec{u}_N(t)\|_V \\ &\leq C_\nu \|\vec{f}(t)\|_H^2 + \frac{1}{4} \nu \|A\vec{u}_N(t)\|_H^2 + \frac{1}{4} \nu \|A\vec{u}_N(t)\|_H^2 + C(\nu) \|\vec{u}_N(t)\|_H^2 \|\vec{u}_N(t)\|_V^4 \end{aligned}$$

Here we have used both the Cauchy and the Young's inequalities. Now we have

$$\frac{d}{dt} \|\vec{u}_N(t)\|_V^2 + \nu \|A\vec{u}_N(t)\|_H^2 \leq 2C_\nu \|\vec{f}(t)\|_H^2 + 2C(\nu) \|\vec{u}_N(t)\|_H^2 \|\vec{u}_N(t)\|_V^4 \quad (3)$$

As in the previous example, $a_N(t) = 2C(\nu) \|\vec{u}_N(t)\|_H^2 \|\vec{u}_N(t)\|_V^2 \in L^1[0, T]$ and then

$$\frac{d}{dt} \|\vec{u}_N(t)\|_V^2 + a_N(t) \|\vec{u}_N(t)\|_V^2 \leq 2C_\nu \|\vec{f}(t)\|_H^2$$

Then the Gronwall inequality is used to conclude that $\{\vec{u}_N(t)\}$ is a bounded sequence in $L^\infty[0, T : V]$. Alternatively, if we integrate (3) from 0 to T, we get

$$\nu \int_0^T \|A\vec{u}_N(t)\|_H^2 dt \leq \|\vec{u}_N(0)\|_V^2 + 2C_\nu \int_0^T \|\vec{f}(t)\|_H^2 dt + 2C(\nu) \int_0^T \|\vec{u}_N(t)\|_H^2 \|\vec{u}_N(t)\|_V^4 dt$$

and since $\{\vec{u}_N(t)\}$ is a bounded sequence in both $L^\infty[0, T : V]$ and in $L^\infty[0, T : H]$, it follows that

$$\nu \int_0^T \|A\vec{u}_N(t)\|_H^2 dt \leq M_3.$$

Finally, since $\|\cdot\|_A$ is equivalent to $\|\cdot\|_{H^2}$, we conclude that $\{\vec{u}_N(t)\}$ is a bounded sequence in $L^2[0, T : H^2(U)^n]$. and in the usual way, we obtain the existence of a weak solution $\vec{u}(t) \in L^2[0, T : H^2(U)^n]$. Note that the (n=2) lemma implies

$$\int_0^T \|B(\vec{u}(t))\|_H^4 dt \leq C \int_1^T \|\vec{u}(t)\|_H^2 \|A\vec{u}(t)\|_H^2 \|\vec{u}_N(t)\|_V^4 dt < \infty$$

Then $B(\vec{u}(t)) \in L^4[0, T : H]$ and since $A\vec{u}(t) \in L^2[0, T : H]$, we have

$$\vec{u}'(t) = \vec{f}(t) - \nu A\vec{u}(t) - B(\vec{u}(t)) \in L^2[0, T : H].$$

In the n=3 case, the equation

$$((\vec{u}'_N(t), \vec{u}_N(t))) + (\nu A\vec{u}_N(t) + B(\vec{u}_N(t)), A\vec{u}_N(t))_H = (\vec{f}(t), A\vec{u}_N(t))_H$$

becomes,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\vec{u}_N(t)\|_V^2 + \nu \|A\vec{u}_N(t)\|_H^2 &\leq \|\vec{f}(t)\|_H \|A\vec{u}_N(t)\|_H + C_1 \|\vec{u}_N(t)\|_V^{3/2} \|A\vec{u}_N(t)\|_H^{3/2} \\ &\leq \frac{1}{4} \nu \|A\vec{u}_N(t)\|_H^2 + \frac{1}{\nu} \|\vec{f}(t)\|_H^2 + \frac{1}{4} \nu \|A\vec{u}_N(t)\|_H^2 + C(\nu)_1 \|\vec{u}_N(t)\|_V^6 \end{aligned}$$

where we have used again, the Cauchy and Young's inequalities. Then

$$\frac{d}{dt} \|\vec{u}_N(t)\|_V^2 + \nu \|A\vec{u}_N(t)\|_H^2 \leq \frac{2}{\nu} \|\vec{f}(t)\|_H^2 + 2C(\nu) \|\vec{u}_N(t)\|_V^6.$$

The term $\|\vec{u}_N(t)\|_V^6$ appearing on the right side of this estimate presents an obstacle which is not easily overcome. We are only able to obtain local in time versions of the results that we found in the case $n=2$.