Summary of Semigroup Results

A is closed and densely defined

\[
(\lambda I + A)^{-1} \text{ exists for all } \lambda > 0
\]

\[
\| (\lambda I + A)^{-1} \|_{L(H)} \leq \lambda^{-1} \text{ for all } \lambda > 0
\]

\[\text{Hille-Yosida properties}\]

H-Y properties \(\implies\) \(\forall u \in H\) and \(\forall t \geq 0\), \(S_n(t)u\) converges strongly in \(H\) to \(S(t)u\).

\(S(t) = \lim_n S_n(t)\) has the following properties:

\[
S(t)\text{ is strongly continuous in } t, t \geq 0
\]

\[
S(0) = I
\]

\[
S(t + s) = S(t) \circ S(s) \quad s, t \geq 0
\]

\[
\| S(t) \|_{L(H)} \leq 1
\]

\[\text{S(t) is a } C^0 - \text{s/g of contractions}\]

Also

\[
S(t) : D_A \rightarrow D_A
\]

\[
AS(t)u = S(t)Au \quad \forall u \in D_A
\]

\[
S'(t)u = -AS(t)u = -S(t)Au \quad \forall u \in D_A
\]

\[
\lim_{t \rightarrow 0} \frac{S(t) - I}{t} u = -Au \quad \forall u \in D_A
\]

\[
S(t)u - u = -\int_0^t S(\tau)Au d\tau \quad \forall u \in D_A
\]

Then

\[\text{H-Y properties } \implies \text{ -A generates a } C^0 - \text{s/g of contractions}\]

Finally, if \(-A\) generates the \(C^0 - \text{s/g of contractions}, S(t), \) then \(-(\lambda I + A)\) generates the semigroup, \(J(t) = e^{-\lambda t}S(t)\) and:

\[
u = \int_0^\infty e^{-\lambda t}S(\tau)(\lambda I + A)u d\tau \quad \forall u \in D_A
\]

\[
(\lambda I + A)^{-1}v = \int_0^\infty e^{-\lambda t}S(\tau)v d\tau \quad \forall v \in H
\]

It follows from these last two results that

\[-A \text{ generates a } C^0 - \text{s/g of contractions } \implies \text{ H-Y properties}\]

We also have
This shows that and

\[ A \text{ is accretive} \]
\[ I + A : D_A \to H \text{ is onto} \]

\( \) Lumer-Phillips properties

and

\( \text{L-P properties } \Rightarrow -A \text{ generates a } C^0 - s/g \text{ of contractions} \)

If \(-A\) generates a \( C^0 - s/g \) of contractions, \( S(t) \), then

\[ \forall u_0 \in D_A \text{ and } \forall f \in C^1(0,T : H) \]
\[ u(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau)d\tau \in C[0,T : D_A] \cap C^1(0,T : D_A) \]
solves

\[ u'(t) + Au(t) = f(t), \quad 0 < t < T \quad \text{and} \quad u(0) = u_0 \]

**Examples**

1. Let \( H = L_2(0,\infty) = H^0(R_+) \), \( A = -a \partial_x \), and \( D_A = H^1(0,\infty) \) for \( a > 0 \).
   Note that \( u \in D_A \) implies \( u(x) \to 0 \) as \( x \to \infty \). Then
   \[ (Au,u)_H = -a \int_0^\infty u'(x)u(x)dx = -\frac{a}{2}u(x)^2|_0^\infty = \frac{a}{2}u(0)^2 \geq 0 \]

Then \( A \) is accretive for \( a > 0 \), and for arbitrary \( v \in H \)

\[ (I + A)u = u(x) - au'(x) = v(x) \]

has the solution

\[ u(x) = \int_x^\infty e^{-\frac{1}{a}(x-y)}v(y)dy \]

This shows that \( I + A \) is onto (i.e., \( R(I + A) = H \)). Then by the L-P theorem, \(-A\) generates a \( C^0 - s/g \), \( S(t) \) and the unique solution of

\[ u'(t) + Au(t) = \partial_t u(x,t) - a\partial_x u(x,t) = f(x,t), \quad u(x,0) = u_0(x) \in D_A; \]

is given by,

\[ u(x,t) = S(t)u_0(x) + \int_0^t S(t-\tau)f(x,\tau)d\tau = u_0(x + at) + \int_0^t f(x + a(t-\tau),\tau)d\tau \]

In order to have this solution it is sufficient to suppose \( u_0 \in D_A = H^1(0,\infty) \) and \( f \) is \( C^1 \) in \( t \) and \( L_2 \) in \( x \).

2. Let \( H = L_2(0,\infty) = H^0(R_+) \), \( A = a\partial_x \), and \( D_A = \{ u \in H^1(0,\infty) : u(0) = 0 \} \).
   Note that \( u \in D_A \) implies \( u(x) \to 0 \) as \( x \to \infty \). Then
   \[ (Au,u)_H = a \int_0^\infty u'(x)u(x)dx = \frac{a}{2}u(x)^2|_0^\infty = 0 \]
Then $A$ is accretive (for any real $a$) and for arbitrary $v \in H$

$$(I + A)u = u(x) + au'(x) = v(x) \quad \text{for } v \in H$$

has the solution

$$u(x) = \int_0^x e^{-\frac{1}{a}(x-y)}v(y)dy.$$ 

Note that it is necessary to have $a > 0$ in order to get $u \in D_A$ for $v \in H$ since a negative value for $a$ would produce an exponentially growing $u(x)$ which is not even in $H$ much less in $D_A$. This shows that $R(I + A) = H$ for $a > 0$. Then $-A$ generates a $C^0 - s\|g\|$, $S(t)$ and $u(t) = S(t)u_0$ is the unique solution of

$$u'(t) + Au(t) = \partial_t u(x,t) + a\partial_x u(x,t) = 0, \quad u(0,t) = 0, \quad u(x,0) = u_0(x) \in D_A;$$

i.e,

$$u(x,t) = u_0(x-at)H(x-at) = S(t)u_0(x).$$

3. Let $H = L^2(R) = H^0(R), \quad A = \partial_x$, and $D_A = H^1(R)$. Note that $u \in D_A$ implies $u(x) \to 0$ as $x^2 \to \infty$. Then

$$(Au,u)_H = \int_{-\infty}^{\infty} u'(x)u(x)dx = \frac{1}{2}u(x)^2|_{-\infty}^a = 0.$$ 

so $A$ is accretive. Note that $-A$ is also accretive in this case. For arbitrary $v \in H$

$$(I \pm A)u = u(x) \pm u'(x) = v(x) \quad \text{for } v \in H$$

implies via Fourier transformation that

$$(1 \pm ia)U(\alpha) = V(\alpha)$$

Then

$$U(\alpha) = (1 \mp ia)\frac{V(\alpha)}{1 + \alpha^2} = W(\alpha) \mp (ia)W(\alpha)$$

and

$$u(x) = w(x) \mp w'(x), \quad w(x) = T_F\left[\frac{V(\alpha)}{1 + \alpha^2}\right] = \int_R e^{-|\alpha|y}v(y)dy$$

Then the ODE has the solution

$$u(x) = \int_0^x e^{-|\alpha|y}v(y)dy \mp \frac{d}{dx}\left(\int_0^x e^{-|\alpha|y}v(y)dy\right)$$

which shows that $R(I \pm A) = H$. Then $-A$ generates a $C^0 - s\|g\|$, $S(t)$ and $u(t) = S(t)u_0$ is the unique solution of

$$u'(t) + Au(t) = \partial_t u(x,t) + \partial_x u(x,t) = 0, \quad u(x,0) = u_0(x) \in D_A;$$

i.e,

$$u(x,t) = u_0(x + t) = S(t)u_0(x).$$

But $+A$ also generates a $C^0 - s\|g\|$, $Z(t)$ and then $u(t) = Z(t)u_0$ is the unique solution of

$$u'(t) - Au(t) = \partial_t u(x,t) - \partial_x u(x,t) = 0, \quad u(x,0) = u_0(x) \in D_A;$$

i.e,
\[ u(x,t) = u_0(x-t) = Z(t)u_0(x) = S(-t)u_0(x). \]

So in this case, since both \( A \) and \(-A\) are accretive and \((I \pm A)\) is onto, there are two \( C^0 - s\)-s. \( S(t) \) and \( S(-t) \). Moreover, by the semigroup property, \( S(t) \circ S(-t) = S(0) = I \), which is to say \( S(t)^{-1} = S(-t) \); i.e., \( S(t) \) forms a \( C^0 \)-group for \( t \in R \).

4. Let \( H = L^2(U) = H^0(U) \), \( A = -\nabla^2 \), and \( D_A = H^1_0(U) \cap H^2(U) \). Then

\[
(Au, u)_H = -\int_U u \nabla^2 u = \int_U |\nabla u|^2 \geq 0 \quad u \in D_A
\]

So \( A \) is accretive. For arbitrary \( f \in H \), the elliptic problem

\[
(\lambda I + A)u = \lambda u - \nabla^2 u = f \quad \text{in } U, \quad u = 0 \quad \text{on } \partial U
\]

has a unique weak solution \( u \in D_A \) since

\[
b[u, v; \lambda] = \int_U \nabla u \cdot \nabla v + \lambda uv
\]

satisfies

\[
b[u, u; \lambda] = \|\nabla u\|^2_0 + \lambda \|u\|^2_0 \quad u \in D_A
\]

and, in particular,

\[
b[u, u; 1] = \|u\|^2_1
\]

which implies \( b[u, u; 1] \) is coercive and \((I + A)\) is then an isomorphism from \( D_A \) onto \( H \). Then \(-A\) generates a \( C^0 - s\)-s, \( S(t) \) and \( u(t) = S(t)u_0 \) is the unique solution of

\[
u'(t) + Au(t) = \partial_t u(x,t) - \nabla^2 u(x,t) = 0,
\]

\[u = 0 \quad \text{on } \partial U \times (0,T)
\]

and \( u(x,0) = u_0(x) \in D_A \);

i.e.,

\[
u(x,t) = \sum_n (u_0, w_n)_H e^{-\lambda_n t} w_n(x) = S(t)u_0(x).
\]

where \( \{w_n\} \) are the family of orthonormal eigenfunctions associated with

\[-\nabla^2 w(x) = \lambda w(x) \quad x \in U, \quad w \in D_A.
\]

5. Consider the problem

\[
\partial_t u(x,t) - \partial_{xx} u(x,t) = 0, \quad 0 < x < 1, \quad t > 0
\]

\[u(x,0) = f(x), \quad 0 < x < 1,
\]

\[\partial_t u(x,0) = g(x) \quad t > 0,
\]

\[u(0,t) = u(1,t) = 0, \quad 0 < x < 1.
\]

Let \( u_1 = \partial_x u \) and \( u_2 = \partial_{tt} u \)

Then \( \partial_{tt} u_1 = \partial_{xx} u = \partial_{xx} u_2 \)

\[\partial_t u_2 = \partial_{tt} u = \partial_{xx} u_1 \]

i.e.,
\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
\end{bmatrix}
- \begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
\end{bmatrix}(x, 0) = \begin{bmatrix}
  f'(x) \\
  g(x) \\
\end{bmatrix}
\]

i.e.,

\[
\partial_t \tilde{U}(t) + A \tilde{U}(t) = 0, \quad \tilde{U}(0) = \tilde{U}_0
\]

where

\[
H = L^2(0, 1), \quad A = \begin{bmatrix}
  0 & 1 \\
  1 & 0 \\
\end{bmatrix}
\partial_x
\]

\[
D_A = \{ \tilde{U} \in H : u_1 \in H^1(0, 1), \ u_2 \in H^1_0(0, 1) \}
\]

Then

\[
(A \tilde{U}, \tilde{U})_H = -\int_0^1 (\partial_x u_2 \cdot u_1 + u_2 \cdot \partial_x u_1) dx = -\int_0^1 d/dx(u_1u_2) dx
\]

\[
= -(u_1u_2)|_{x=0} = 0 \quad \text{(since } u_2 \in H^1_0(0, 1)\text{)}
\]

This proves A is accretive, in fact, conservative. Now for \( \lambda \neq 0, \tilde{F} \in H\), consider

\[
\lambda \begin{bmatrix}
  u_1 \\
  u_2 \\
\end{bmatrix} + A \begin{bmatrix}
  u_1 \\
  u_2 \\
\end{bmatrix} = \begin{bmatrix}
  \lambda u_1 - \partial_x u_2 \\
  \lambda u_2 - \partial_x u_1 \\
\end{bmatrix} = \begin{bmatrix}
  F_1 \\
  F_2 \\
\end{bmatrix}
\]

Then

\[
\lambda \partial_x u_1 - \partial_x u_2 = \partial_x F_1 \quad \text{and} \quad \partial_x u_1 = \lambda u_2 - F_2,
\]

or

\[
-\partial_x u_2 + \lambda^2 u_2 = \partial_x F_1 + \lambda F_2
\]

Since \( \partial_x F_1 + \lambda F_2 \in H^1(0, 1)\), this last equation has a unique weak solution \( u_2 \in H^1_0(0, 1)\), by the previously developed elliptic theory. Then

\[
\lambda u_1 = F_1 + \partial_x u_2 \in L^2(0, 1), \quad \partial_x u_1 = \lambda u_2 - F_2 \in L^2(0, 1),
\]

so \( u_1 \in H^1(0, 1)\) and \( \tilde{U} \in D_A \). This shows that \( \lambda + A : D_A \rightarrow H \) is surjective for all \( \lambda \neq 0 \).

Then \( A \) generates a \( C^0 - s/g\), actually both \( \pm A \) generate \( C^0 - s/g \)'s so \( A \) generates a group of solution operators, \( G(t) \).

This group \( G(t) \), (using a previously discovered solution to the wave equation) is seen to be given by

\[
G(t)[\tilde{U}(x, 0)] = \begin{bmatrix}
  \frac{1}{2} (\tilde{f}'(x + t) + \tilde{f}'(x - t)) + \frac{1}{2} (\tilde{g}(x + t) - \tilde{g}(x - t)) \\
  \frac{1}{2} (\tilde{f}'(x + t) - \tilde{f}'(x - t)) + \frac{1}{2} (\tilde{g}(x + t) + \tilde{g}(x - t))
\end{bmatrix}
\]

where \( \tilde{f}, \tilde{g} \) denote the odd 2-periodic extensions of \( f \) and \( g \).