Introduction to Semigroups for Evolution Equations

1. Abstract Initial Value Problems

We have considered initial boundary value problems (IBVP’s) of the form

\[
\begin{align*}
\partial_t u(x,t) + Lu(x,t) &= f(x,t) & x \in U, & t \in (0,T) \\
u(x,0) &= u_0(x) & x \in U, \\
u(x,t) &= 0, & x \in \partial U, & 0 < t < T,
\end{align*}
\]

where \(L\) denotes an elliptic operator of order 2. This problem can be written abstractly as

\[
\begin{align*}
\frac{d}{dt}u(t) + Au(t) &= f(t), & t \in (0,T) \\
u(0) &= u_0,
\end{align*}
\]

where

\[(Au, v)_H = B[u,v], \quad \forall u, v \in V \subset H \subset V'.\]

We showed that if the bilinear form \(B\) is coercive then the linear mapping \(A\) is an isomorphism from \(V\) onto \(V'\). Recall that there exists a family of \(H\)-orthonormal eigenfunctions for \(A\) such that

\[\|Aw_k\|_H = |\lambda_k| \|w_k\|_H = |\lambda_k| \to \infty \text{ as } k \to \infty.\]

Then \(A\) is clearly not bounded as a mapping from \(H\) into \(H\). If we define

\[D_A = \{u \in H : u \in V\},\]

then

\[H^1_0(U) \cap H^2(U) \subset D_A \quad \text{and} \quad A : D_A \to H.\]

Here we are saying that \(A\) restricted to \(D_A\) takes its values in \(H\) rather than \(V'\) but \(A\) is still not bounded in the norm of \(H\). Note, however, that if

\[\{u_n\} \subset D_A, \quad u_n \to u \text{ in } H, \quad \text{and} \quad Au_n \to v \text{ in } H,\]

then

\[u_n \to u \text{ in } D'(U), \quad \text{and} \quad Au_n \to v \text{ in } D'(U).\]

This means \(v = Au\) in the sense of distributions which implies in turn that

\[(Au, \phi) = (v, \phi) \quad \forall \phi \in D(U)\]

and since the test functions are dense in \(H\), it follows that \(v = Au\) in \(H\), so \(u \in D_A\) and \(Au = v\). Any operator with the property that

\[
\left\{ \begin{array}{l}
\{u_n\} \subset D_A, \quad u_n \to u \text{ in } H, \\
\text{and} \quad Au_n \to v \text{ in } H
\end{array} \right\} \implies \{u \in D_A \text{ and } Au = v\}
\]

is said to be a closed operator. Clearly any bounded/continuous linear operator is closed but, as we have just seen, the converse is, in general false. In our example, \(H^1_0(U) \cap H^2(U) \subset D_A\) implies that \(D_A\) must be dense in \(H = L^2(U)\).

Typically, we will consider problems

\[
\begin{align*}
\frac{d}{dt}u(t) + Au(t) &= f(t), & t \in (0,T) \quad (IVP) \\
u(0) &= u_0,
\end{align*}
\]

with the assumption that \(A\) is closed and densely defined on \(H\). We will also suppose
If $A$ and $B$ are bounded linear operators on Hilbert space $H$ then
\[(Ax,x)_H \geq 0 \quad \forall x \in D_A \subset H.\]

Then $A$ is said to be accretive. Note that this is not the same as coercive. For example,
\[A = -\nabla^2 \text{ on } D_A = H^1_0(U) \cap H^2(U)\]
satisfies
\[(Au,u)_H = -\int_U u\nabla^2 u \, dx = \int_U \nabla u \cdot \nabla u \, dx \geq 0 \quad \forall u \in D_A\]
i.e., the Laplacian is both coercive and accretive. More generally, if $A$ is any elliptic operator of order 2, we can change the dependent variable in the initial value problem to obtain a new problem with $A$ replaced by $A + \mu I$, which is coercive for $\mu$ sufficiently large. Then all parabolic IBVP’s can be assumed without loss of generality to involve an operator $A$ that is coercive and hence accretive. On the other hand,
\[A = -\frac{d}{dt} \text{ on } D_A = H^1(0,1)\]
satisfies
\[(Au,u)_H = -\int_0^1 u'(x)u(x) \, dx = -\int_0^1 \frac{1}{2} \frac{d}{dx} u^2(x) \, dx = \frac{1}{2} [u(0)^2 - u(1)^2] = 0\]
so this operator is accretive but not coercive.

**Lemma 1** If $A$ is closed, densely defined and accretive, then (IVP) has at most one solution.

Proof- Suppose $u(t)$ solves (IVP) with $u_0 = 0 = f$. Then
\[\frac{d}{dt} \|u(t)\|_H^2 = 2(u'(t),u(t))_H = -2(Au(t),u(t))_H \leq 0,\]
hence
\[\|u(t)\|_H \leq \|u(0)\|_H = 0 \quad \text{for all } t > 0.\]

Note that if $u(t)$ solves (IVP) for an accretive operator $A$, then it is necessarily the case that $\|u(t)\|_H \leq \|u(0)\|_H$.

2. The Linear Space $L(H)$

If $A$ and $B$ are bounded linear operators on Hilbert space $H$ then
\[(aA + bB)(x) = aA(x) + bB(x) \quad \forall x \in H\]
defines another bounded linear operator on $H$; i.e., $L(H)$ is a linear space. We can define a norm on $L(H)$ by
\[\|A\|_{L(H)} = \sup \{ \|Ax\|_H : \|x\|_H = 1 \}.\]

Then it is an exercise to show that $L(H)$ is a complete normed linear space for this norm; i.e., if $\{A_n\}$ is a Cauchy sequence in $L(H)$ then there exists an $A$ in $L(H)$ such that $\|A_n - A\|_{L(H)} \to 0$ as $n \to \infty$. We say in this case that $A_n$ converges in the operator norm to the limit, $A$. A sequence $\{A_n\} \subset L(H)$ such that for some $A \in L(H)$ we have $\|A_n x - Ax\|_H \to 0$ as $n \to \infty \forall x \in H$, is said to converge strongly to the limit $A$. Convergence in the operator norm implies strong convergence but not conversely.
We now state two lemmas that will be needed later.

**Lemma 2 (Neumann Series)** If $B \in L(H)$ with $\|B\|_{L(H)} < 1$, then $(I - B)^{-1} \in L(H)$ and

$$(I - B)^{-1} = \sum_{n=0}^{\infty} B^n$$

The proof of this result is a complete analogue of the proof of the formula for the sum a convergent geometric series, where absolute values are replaced by operator norms.

**Lemma 3** Suppose $A : D_A \to H$ and $(\mu I - A)^{-1} \in L(H)$ for some complex number, $\mu$. Then for $\lambda \in C$, $(\lambda I - A)^{-1} \in L(H)$ if and only if $[I - (\mu - \lambda)(\mu I - A)^{-1}]^{-1} \in L(H)$ and in this case

$$(\lambda I - A)^{-1} = (\mu I - A)^{-1} [I - (\mu - \lambda)(\mu I - A)^{-1}]^{-1}$$

**Proof** Let $B = I - (\mu - \lambda)(\mu I - A)^{-1}$. If $B^{-1} \in L(H)$ then

$$(\lambda I - A)(\mu I - A)^{-1} B^{-1} = (\lambda I - \mu I + \mu I - A)(\mu I - A)^{-1} B^{-1} = (\lambda - \mu)(\mu I - A)^{-1} + I) B^{-1} = I \quad \text{on } H$$

and

$$(\mu I - A)^{-1} B^{-1}(\lambda I - A) = (\mu I - A)^{-1} B^{-1}((\lambda - \mu I + \mu I - A) = (\mu I - A)^{-1} B^{-1}[(\lambda - \mu)(\mu I - A)^{-1} + I](\mu I - A) = I \quad \text{on } D_A.$$ 

The converse is proved similarly.

3. **Semigroups of Solution Operators**

We return to the initial value problem

$$u'(t) + Au(t) = 0, \quad t \in (0, T) \quad \text{ (IVP)}$$

$$u(0) = u_0,$$

assuming that $A$ is closed, accretive and densely defined on $H$. Suppose that for every $u_0 \in D_A$ there exists a unique solution $u(t)$, and let us denote the dependence of $u(t)$ on $u_0$ by writing $u(t) = S(t)[u_0]$. Then, integrating the differential equation shows

$$u(t) = S(t)[u_0] = u_0 + \int_0^t -Au(\tau) \, d\tau.$$

It is evident that $S(t) : H \to H$ defines a linear operator on $H$. In addition, $\forall u_0 \in D_A$

i) $S(0)[u_0] = u_0$; i.e., $S(0) = I$

ii) $S(t + \tau)[u_0] = S(t)[S(\tau)[u_0]] = S(\tau)[S(t)[u_0]]$;

i.e.,

$$(t + \tau) = S(t) \circ S(\tau) = S(\tau) \circ S(t) \text{ for } t, \tau \geq 0.$$ 

The second result follows from the uniqueness of the solution. We summarize these two results by saying that $\{S(t) : t \geq 0\}$ is a semigroup of solution operators on $H$. If, in addition,
then we say that \( S(t) \) is a **strongly continuous semigroup**, a \( C^0 - \text{semigroup} \). Note that since \( A \) is accretive,

\[
\frac{d}{dt} \|u(t)\|_H^2 = 2(-Au(t), u(t))_H \leq 0,
\]

hence,

\[
\|S(t)[u_0]\|_H = \|u(t)\|_H \leq \|u(0)\|_H = \|u_0\|_H;
\]

i.e.,

\[
\|S(t)\|_{L(H)} \leq 1 \quad \forall t \geq 0.
\]

In this case, we say \( \{S(t) : t \geq 0\} \) is a **semigroup of contractions**.

Associated with any \( C^0 - \text{semigroup} \) there is a linear operator, \( B \), called the generator of the semigroup; \( B \) is defined by

\[
D_B = \left\{ x \in H : \lim_{h \to 0} \frac{S(h) - I}{h} x \text{ exists and belongs to } H \right\}
\]

\[
Bx := \lim_{h \to 0} \frac{S(h) - I}{h} x \quad \forall x \in D_B.
\]

It will be shown now that if \( S(t) \) is the semigroup associated with the IVP involving operator \( A \), then \( B \) is an extension of \( -A \). If \( A \) is accretive, then the semigroup generated by \( B \) will be a contraction semigroup.

**Remark:** \( f(t) = e^{at} \) is defined by

- i) \( f(t)f(\tau) = f(t + \tau) \)
- ii) \( f(0) = 1 \)
- iii) \( a = \lim_{h \to 0} \frac{f(h) - 1}{h} \).

i.e., Assuming only that \( f \) is continuous and has the semigroup property implies that \( f \) is differentiable.

**Properties of \( S(t) \).**

**Lemma 4** Suppose \( S(t) \) is a \( C^0 - \text{semigroup} \) and \( u_0 \in D_B \). Then

- i) \( S(t)[u_0] \in D_B \quad \forall t \geq 0 \)
- ii) \( BS(t)[u_0] = S(t)[Bu_0] \quad \forall t \geq 0 \)
- iii) \( (0, \infty) \ni t \mapsto S(t)[u_0] \) is differentiable for \( t > 0 \)

and

\[
\frac{d}{dt} \left( S(t)[u_0] \right) = BS(t)[u_0] \quad \forall t > 0
\]

- iv) \( S(t)[u_0] - u_0 = \int_0^t BS(\tau)[u_0] d\tau = \int_0^t S(\tau)[Bu_0] d\tau. \)

**Proof** For \( u_0 \in D_B \)

\[
\frac{S(h)S(t)[u_0] - S(t)[u_0]}{h} = S(t)\left( \frac{S(h)[u_0] - u_0}{h} \right)
\]
\[
\frac{S(h) - I}{h} S(t)[u_0] = S(t) \frac{S(h) - I}{h} u_0
\]

\[
BS(t)[u_0] = S(t)[Bu_0]
\]

The limit on the right side of this equality exists because we have assumed that \(u_0 \in D_B\). Then the limit on the left side exists, which implies that \(S(t)[u_0] \in D_B\), and, moreover, ii) holds. Note further that

\[
\frac{S(t + h)[u_0] - S(t)[u_0]}{h} = \frac{S(t)S(h)[u_0] - S(t)[u_0]}{h} = \frac{S(h) - I}{h} S(t)[u_0]
\]

\[
\frac{d}{dh} (S(t)[u_0]) \quad \text{(derivative from the right)} = B(S(t)[u_0])
\]

For \(0 < h < t\), we have

\[
\frac{S(t)[u_0] - S(t - h)[u_0]}{h} = \frac{S(h) - I}{h} S(t - h)[u_0]
\]

(derivative from the left) \(\frac{d}{dh} (S(t)[u_0]) = B(S(t)[u_0])\); i.e., for \(u_0 \in D_B\), \(t > 0\), \(S(t)[u_0]\) is differentiable with \(\frac{d}{dh} (S(t)[u_0]) = B(S(t)[u_0])\).

Now integrating both sides of this expression from 0 to \(t > 0\),

\[
S(t)[u_0] - u_0 = \int_0^t B(S(\tau)[u_0]) d\tau = \int_0^t S(\tau)[Bu_0] d\tau \quad \forall u_0 \in D_B
\]  

(1)

**Properties of B**

**Lemma 5**- Suppose B is the generator of a strongly continuous semigroup on H. Then

i) \(D_B\) is dense in \(H\)

ii) B is closed

Proof- For an arbitrary \(x \in H\), let \(x_t = \int_0^t S(\tau)[x] d\tau\). Then

\[
\frac{S(h)[x_t] - x_t}{h} = \int_0^t \frac{S(\tau + h)[x] d\tau - \int_0^t S(\tau)[x] d\tau}{h}
\]

\[
= \int_h^{t+h} S(\tau)[x] d\tau - \int_0^t S(\tau)[x] d\tau
\]

\[
= \frac{\int_0^{t+h} S(\tau)[x] d\tau - \int_0^t S(\tau)[x] d\tau}{h}
\]

and, as \(h \to 0^+\), we get \(Bx_t = S(t)x - x\). Since the limit on the right side exists, it follows that the limit on the left exists, which is to say, \(x_t \in D_B\), for \(t > 0\). Moreover, \(\frac{1}{t} x_t \to x\) in \(H\) as \(t \to 0^+\) hence \(D_B\) is dense in \(H\). Note that we have proved,

\[
S(t)x - x = B \int_0^t S(\tau)[x] d\tau \quad \forall x \in H, \ t \geq 0.
\]  

(2)

To prove B is closed, suppose
Then \( S(t)[u_n] - u_n = \int_0^t (S(r)[Bu_n]) \, dr \) and as \( n \to \infty \) we get,
\[
S(t)[u] - u = \int_0^t (S(r)[v]) \, dr.
\]

But then,
\[
\frac{S(t)[u] - u}{t} = \frac{1}{t} \int_0^t (S(r)[v]) \, dr.
\]

Since the limit on the right side exists, the limit on the left must exist, which implies \( u \in D_B \) with \( Bu = v \); i.e., \( B \) is closed.

We have seen that an initial value problem may generate a semigroup and that associated with any strongly continuous semigroup is a generator. Now we will see the connection between the IVP and the generator of its semigroup.

**Theorem 1**  Suppose \( A : D_A \to H \) is closed and densely defined. Suppose also that for any \( u_0 \in D_A \) there is a unique \( u(t) \) such that

1. \( u \in C^0([0, \infty) : H) \cap C^1((0, \infty) : H) \)
2. \( u'(t) + Au(t) = 0 \quad t > 0 \)
3. \( u(0) = u_0 \)

Then \( u(t) := S(t)[u_0] \) defines a strongly continuous semigroup (of contractions if \( A \) is accretive) on \( H \). The generator, \( B \), of this semigroup is then an extension of \( -A \).

**Proof** - We have already noted that \( u(t) := S(t)[u_0] \) defines a strongly continuous semigroup on \( H \). Let the generator be denoted by \( B \). Then it follows from 1) and 2) that for all \( u_0 \in D_A \),
\[
u(t) = S(t)[u_0] = u_0 + \int_0^t -Au(\tau) \, d\tau.
\]

Then
\[
\frac{S(t)[u_0] - u_0}{t} = \frac{1}{t} \int_0^t -Au(\tau) \, d\tau.
\]

The limit on the right exists by properties of the integral, hence the limit on the left also exists. But then \( u_0 \in D_A \) implies \( u_0 \in D_B \) and \( Bu_0 = -Au_0 \) for all \( u_0 \in D_A \). This proves \( B \) is an extension of \( -A \).

**Examples:**

1) Consider \( u'(t) = bu(t), \quad u(0) = u_0 \) for \( b < 0 \).

Then
\[
u(t) = e^{bt}u_0 = S(t)[u_0],
\]

and
\[
|S(t)[u_0]| = |e^{bt}u_0| \leq |u_0| \quad \text{since} \quad b < 0.
\]
Note that $a = -b > 0$ satisfies $(au, u)_H = au^2 \geq 0 \quad H = \mathbb{R}^1$.

2) Consider $u'(t) + Au(t) = 0, \quad u(0) = u_0$

where

$$A = -\nabla^2 \quad \text{with} \quad D_A = H^1_0(U) \cap H^2(U) \subset H = H^0(U).$$

\text{i.e.}, This is the heat equation with homogeneous Dirichlet boundary conditions and we recall that $A$ is closed, densely defined and accretive. Also, as we have seen often before,

$$u(t) = \sum_{n=1}^{\infty} (u_0, \phi_n)_H e^{-\lambda_n t} \phi_n(x) = S(t)[u_0].$$

In fact if we denote by $T$ the isomorphism

$$H \ni u_0 \rightarrow \{(u_0, \phi_n)_H\} \in \ell_2,$$

then $S(t)[\cdot] = T^{-1}\{e^{-\lambda_n t} T(\cdot)\}$ and $B = -A = \nabla^2$ is the generator of the semigroup.