Problems

Problems from the text:
page 12, problem 3
page 290 problems 5, 6, 7, 8
page 291 problems 10, 13, 14, 16, 17
page 292 problem 18
page 345 1, 2, 3, 4, 7
page 425 1 through 10 (Hint for 2 and 6: expand \( u(x,t) \) in terms of an appropriate family of eigenfunctions \( \{\phi_n(x)\} \))

1. (equivalence of definition of weak derivative)
   Show that for \( u \in L^2(\mathbb{R}^1) \), the following are equivalent:
   \[
   \begin{align*}
   (a) & \; \partial_x u \in L^2(\mathbb{R}^1) \\
   (b) & \; z \dot{u} \in L^2(\mathbb{R}^1) \\
   (c) & \; \frac{u(x+h) - u(x)}{h} \text{ converges in } L^2(\mathbb{R}^1) \text{ as } h \to 0 \\
   (d) & \; \text{there exists a sequence of test functions } \phi_n, \text{ such that } \phi_n \text{ converges to } f \text{ in } L^2(\mathbb{R}^1) \text{ and } \partial_x \phi_n \text{ converges in } L^2(\mathbb{R}^1)
   \end{align*}
   \]

2. For what values of \( s \) is the characteristic function of \( I = [0,1] \) in \( H^s(\mathbb{R}) \)?
   For what values of \( s \) is the characteristic function of \( I^2 = [0,1]^2 \) in \( H^s(\mathbb{R}^2) \)?

3. To which spaces \( H^m(U) \) do the following functions \( u \) belong?
   \[
   (a) \quad u(x) = \begin{cases} 
   0 & \text{if } 0 < x < 1 \\
   x - 1 & \text{if } 1 \leq x < 2 \\
   x^3 - x^2 - 3 & \text{if } 2 \leq x < 3 
   \end{cases}
   \]
   \[
   (b) \quad u(x, y) = \begin{cases} 
   xy & \text{if } 0 < x < 1, 0 < y < 1 \\
   x(2-y) & \text{if } 0 < x < 1, 1 < y < 2 \\
   x & \text{if } 0 < x < .5, y = 1 \\
   4 & \text{if } x = .5, y = 1 \\
   x & \text{if } .5 < x < 1, y = 1 
   \end{cases}
   \]
4) Let \( u(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 2-x & \text{if } 1 < x < 2, \end{cases} \) and \( v(x) = \sin \pi x \).

a) Determine whether \( u \) and \( v \) are orthogonal in \( L^2(0,2) \).

Are they orthogonal in \( H^1(0,2) \)?

b) Find the distance from \( u \) to \( v \) in \( L^2(0,2) \) and in \( H^1(0,2) \).

5. Use the Sobolev embedding theorem to show that the function

\[ u(x,y) = \begin{cases} x^2y^2 & \text{if } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases} \]

is continuous on \( \Omega = (-1,1) \times (-1,1) \).

6. Suppose \( u \in H^4(U) \), \( v \in H^2(U) \). Show that

\[ \int_U \nabla^2 u \nabla^2 v \, dx = \int_U (\nabla^4 u) v \, dx + \int_{\partial U} \left[ (\nabla^2 u) \nabla n - (\nabla^2 u) v \right] \, dS \]

Here \( \nabla n = n \cdot \nabla \) where \( n \) = outward unit normal to \( \partial U \).

7. By evaluating the relevant integrals, show that:

- \( x \) and \( x \ln x \) \( \in H^1(0,1) \),
- \( \ln x \) \( \in L^2(0,1) \) but \( x^{-1} \notin L^2(0,1) \)
- \( x^{-1} \) \( \in H^{-1}(0,1) \) (a function belongs to \( H^{-1}(0,1) \) if it is the weak derivative of an element of \( L^2(0,1) \)).

8. Suppose \( u(x) \) is continuous for all \( x \) in \( R \). Show that for all test functions \( \phi(x) \),

\[ \lim_{h \to 0} \frac{\int \phi(x)[u(x+h) - u(x)] \, dx}{h} = -\int u(x) \phi'(x) \, dx \]

9. Let \( U = (a,b) \) denote a bounded open set in \( R \).

a) Show that if \( u \in H^1_0(U) \), then \( u \) is absolutely continuous and \( u(a) = u(b) = 0 \).

b) Show that \( H^1(U) \) is the direct sum of \( H^1_0(U) \) and the space of functions \( v(x) = Ae^x + Be^{-x} \) for arbitrary constants \( A, B \); i.e., show that any \( v \) of this form is orthogonal to every function in \( H^1_0(U) \).

c) Show that for every \( u \) in \( C^0_0(U) \),

\[ u(x)^2 = \int_a^x 2u'(z)u(z) \, dz \]

and then use the C-S inequality to show that for some constant \( C > 0 \),
\[
\int_a^b u(z)^2dz \leq C \int_a^b u'(z)^2dz
\]

d) Show that the previous inequality continues to hold for all \( u \) in \( H^1_0(U) \).

10. Let
\[
\begin{align*}
    u_1(x) &= \begin{cases} 
        1 - |x| & \text{if } |x| < 1 \\
        0 & \text{if } |x| > 1
    \end{cases} \\
    u_2(x) &= \begin{cases}
        x^2 & \text{if } |x| < 1, \\
        0 & \text{if } |x| > 1
    \end{cases}
\end{align*}
\]
Find \( u'_1(x) \), the weak derivative of \( u_1(x) \) and show that it is a locally integrable function. Show that the weak derivative of \( u_2(x) \) is not locally integrable.

11. Show that for a bounded linear operator \( A \) on \( H \), if \( \|Ax\|_H \geq c\|x\|_H \) for all \( x \) in \( H \), then the range of \( A \) is a closed subset of \( H \).

12. Suppose \( u(x,y) \in H^m(R^2_+) \) and let
\[
v(x,y) = \begin{cases} 
    u(x,y) & \text{if } y > 0 \\
    \sum_{k=1}^m a_k u(x,-ky) & \text{if } y < 0
\end{cases}
\]
where
\[
\sum_{k=1}^m (-k)^s a_k = 1 \quad \text{for } 0 \leq s \leq m - 1.
\]
Show that
\[
\partial_x^s \partial_y^q v(x,y) = \begin{cases} 
    \partial_x^s \partial_y^q u(x,y) & \text{if } y > 0 \\
    \sum_{k=1}^m (-k)^s a_k \partial_x^s \partial_y^q u(x,-ky) & \text{if } y < 0
\end{cases}
\]
and that this implies
\[
\lim_{y \to 0^+} \partial_x^s \partial_y^q v(x,y) = \lim_{y \to 0^+} \partial_x^s \partial_y^q u(x,y) \quad \text{for } 0 \leq q \leq m - 1.
\]
Does this imply that the mapping defined by \( v(x,y) = Eu(x,y) \) maps \( H^m(R^2_+) \) continuously into \( H^m(R^2_+) \)?