The Trace and Embedding Theorems for a General Bounded Open Set

Now we show how the primitive versions of the results we have proved (i.e., when \( U = \mathbb{R}^n \)) can be used to deduce analogous results when \( U \) is a more general open set. We will describe now the special properties \( U \) must have if this extension of results is to work.

1. Flattening the Boundary

Suppose \( U \) is a bounded open set in \( \mathbb{R}^n \). Then \( U \) is said to be regular if:

- \( p_k \in C^m(\mathbb{R}^n) \) maps \( O_k \) onto open set \( Q \)
- \( q_k = p_k^{-1} \in C^m(\mathbb{R}^n) \) maps \( Q \) onto \( O_k \)
- \( p_k : O_k \cap U \to Q^+ = Q \cap \{ y_n > 0 \} \)
- \( p_k : O_k \cap \partial U \to Q_0 = Q \cap \{ y_n = 0 \} \)

Then \( \partial U \subset \bigcup_{k=1}^{M} O_k \) and we suppose also that \( \bar{U} \subset \bigcup_{k=0}^{M} O_k \), where \( O_0 \) denotes an open set in the interior of \( U \). This property of having a boundary that "looks like" \( \mathbb{R}^{n-1} \) near each of its points is what will allow us to extend our primitive versions of results to \( U \). We need one more devise to make this argument work.

We define a partition of unity subordinate to the open covering \( \{O_k : 0 \leq k \leq M\} \). This is a set of functions \( a_k(x) \in C_c^\infty(\mathbb{R}^n) \) such that

a. \( \text{supp} a_k \subset O_k \) (then \( a_0 \in C_c^\infty(U) \))

b. \( a_k \geq 0 \) on \( O_k \)

c. \( \sum_{k=0}^{M} a_k(x) = 1 \) \( \forall x \in U \)

Then for \( f \in H^m(U) \), we can write

\[
f(x) = a_0(x)f(x) + \sum_{k=1}^{M} a_k(x)f(x)\]
Lemma 2. \textit{Proof-} Recall the definition, for $g \in H^m(U)$ then
\[
\frac{\partial g}{\partial x} = \sqrt{a_0(x)} \left( f \sqrt{a_0} \right)(x) + \sum_{k=1}^{M} \sqrt{a_k(x)} q_{k}^{\#}(p_{k}^{\#}(f \sqrt{a_k})) (x)
\]

where
\[
H^m(U) \ni f \to p_{k}^{\#} f = f(q(y)) \in H^m(Q_+)
\]
\[
H^m(Q_+) \ni g \to q_{k}^{\#} g = g(p(x)) \in H^m(U).
\]

Then
\[
f \sqrt{a_0} \in H^m_0(U) \quad \text{and} \quad Z(f \sqrt{a_0}) \in H^m(R^n),
\]
\[
p_{k}^{\#}(f \sqrt{a_k}) \in H^m(R^n_+) \quad \text{with} \quad \text{supp} p_{k}^{\#}(f \sqrt{a_k}) \subset Q_+.
\]

Now define $A : H^m(U) \to H^m(R^n) \times H^m(R^n_+)^M$ as
\[
Af = \{ f \sqrt{a_0}, p_{1}^{\#}(f \sqrt{a_1}), \ldots, p_{M}^{\#}(f \sqrt{a_M}) \}
\]
and $B : H^m(R^n) \times H^m(R^n_+)^M \to H^m(U)$ as
\[
B[v_0, \ldots, v_M] = v_0 \sqrt{a_0} + \sum_{k=1}^{M} \sqrt{a_k} q_{k}^{\#}(v_k).
\]

Then $B[Af] = f \quad \forall f \in H^m(U)$.

Evidently, $A$ decomposes $f \in H^m(U)$ into $M + 1$ pieces, one of which lives on $O_0$ and $M$ others living on the sets $O_k \cap U$, $1 \leq k \leq M$. The mapping $B$ reassembles these pieces into the original function, $f$.

Similarly, define $A' : H^m(\partial U) \to H^m(R^{n-1})^M$ and $B' : H^m(R^n_+)^M \to H^m(U)$ by
\[
A'f = \{ p_{1}^{\#}(f \sqrt{a_1}), \ldots, p_{M}^{\#}(f \sqrt{a_M}) \}
\]
\[
B'[v_1, \ldots, v_M] = \sum_{k=1}^{M} \sqrt{a_k} q_{k}^{\#}(v_k).
\]

These two mappings deal only with functions living on the boundary of $U$ so $A'$ decomposes $f \in H^m(\partial U)$ into $M$ pieces, living on the sets $O_k \cap U$, $1 \leq k \leq M$. The mapping $B'$ reassembles these pieces into the original function, $f$.

2. Basic Extension Lemma

Lemma 2.1 (Basic Extension lemma) For $U$ a bounded, open and regular set in $R^n$, every $u \in H^m(U)$ can be extended to $\tilde{u} \in H^m_0(V)$ for $U \subset V$.

Proof- Recall the definition, for $u \in H^m(R^n_+)$,
\[
E_1 u(x', x_n) = \begin{cases} 
      u(x', x_n) & \text{if } x_n > 0 \\
      a(x_n) Eu(x', x_n) & \text{if } x_n < 0
\end{cases}
\]

where
\[ a(x_n) \in C^\infty(R^n), \quad a(x) = \begin{cases} 1 & \text{if } x_n > 0 \\ 0 & \text{if } x_n < -1 \end{cases} \]

Then \( E_1u \in H^m(R^n) \) for all \( u \in H^m(R^n) \) where \( R^n_1 = \{ (x', x_n) : x' \in R^{n-1}, \ -1 < x_n < \infty \} \). This modified extension operator smoothly extends the function \( u \in H^m(R^n) \) to a neighborhood of the boundary of \( R^n_1 \). Now for \( U \subset V \) we have

\[
\begin{align*}
H^m(U) & \longrightarrow A \longrightarrow H^m(R^n) \times H^m(R^n_1)^M \\
\downarrow \text{id} & \downarrow E_1 \\
H_0^m(V) & < -B - - - H^m(R^n) \times H^m(R^n_1)^M \\
\downarrow Z & \\
H^m(R^n) & \longrightarrow C^k(R^n) \rightarrow C^k(U)
\end{align*}
\]

i.e., \( E_1 \) extends each function in \( H^m(R^n) \) smoothly to a function in \( H^m(R^n_1) \). Since \( R^n_1 \subset R^n_2 \), applying the mapping \( B \) produces a smooth function with support in an open neighborhood of \( U \).

\section*{3. Trace and Embedding Theorems for a General Open Set}

\textit{Theorem 3.1 Sobolev Embedding Theorem}

For \( U \) a bounded, open and regular set in \( R^n \), every \( u \in H^m(U) \) can be identified with \( \tilde{u} \in C^k(\bar{U}) \), for \( m > k + \frac{n}{2} \):

i.e., \( e : H^m(U) \hookrightarrow C^k(\bar{U}) \), for \( m > k + \frac{n}{2} \) is a continuous injection.

Proof -

\[
\begin{align*}
H^m(U) & \longrightarrow A \longrightarrow H^m(R^n) \times H^m(R^n_1)^M \\
\downarrow \text{id} & \downarrow E_1 \\
H_0^m(V) & < -B - - - H^m(R^n) \times H^m(R^n_1)^M \\
\downarrow Z & \\
H^m(R^n) & \longrightarrow C^k(R^n) \rightarrow C^k(U)
\end{align*}
\]

where \( H^m(R^n) \longrightarrow C^k(R^n) \) denotes the continuous injection of theorem 2.1.

and \( C^k(R^n) \longrightarrow C^k(\bar{U}) \) denotes the restriction from \( R^n \) to \( U \).

\textit{Theorem 3.2 Rellich Embedding Theorem}

For \( U \) a bounded, open and regular set in \( R^n \), the embedding of \( H^m(U) \) into \( H^{m-1}(U) \) is compact; i.e. any sequence that is bounded in the norm of \( H^m(U) \) contains a subsequence that is convergent in the norm of \( H^{m-1}(U) \).

Proof -

\[
\begin{align*}
H^m(U) & \longrightarrow A \longrightarrow H^m(R^n) \times H^m(R^n_1)^M \\
\downarrow \text{id} & \downarrow E_1 \\
H_0^m(V) & < -B - - - H^m(R^n) \times H^m(R^n_1)^M \\
\downarrow e & \\
e & \longrightarrow H^{m-1}(U)
\end{align*}
\]

where \( e : H_0^m(V) \longrightarrow H^{m-1}(V) \).
denotes the compact embedding of the corollary to theorem 2.2 and
\[ H^{m-1}(V) \to H^{m-1}(U) \]
denotes the restriction from \( V \) to \( U \),

**Theorem 3.3 The Smooth Approximation Theorem**
For \( U \) a bounded, open and regular set in \( \mathbb{R}^n \), \( C^\infty(\overline{U}) \) is dense in \( H^m(U) \).

Proof-

\[
\begin{align*}
C^\infty(\overline{U}) & \to A \to H^m(\mathbb{R}^n) \times H^m(\mathbb{R}^n)_j^M \\
\downarrow \text{id} & \downarrow E_1 \\
U \subseteq V & \quad C^0_U(V) \to B \to H^m(\mathbb{R}^n) \times H^m(\mathbb{R}^n)_j^M \\
\downarrow i & \\
H^0_U(V) & \to H^m(U),
\end{align*}
\]

where
\[
C^\infty_U(V) \to C^0_U(V) \to H^m(U)
\]
is an injection with a dense image since \( C^\infty_U(V) \) is dense in both \( C^0_U(V) \) and \( H^0_U(V) \) so it follows that \( C^0_U(V) \) is dense in \( H^0_U(V) \) and
\[
H^0_U(V) \to H^m(U)
\]
denotes restriction from \( V \) to \( U \).

**Theorem 3.4 The Trace Theorem.**
For \( U \) a bounded, open and regular set in \( \mathbb{R}^n \),
\[
T_j : H^m(U) \to H^{m-j/2}(\partial U) \quad 0 \leq j \leq m - 1,
\]
is a continuous linear surjection and \( T_ju = 0 \) if and only if \( u \in H^m_U(V) \).

Proof

\[
\begin{align*}
H^m(U) & \to A \to H^m(\mathbb{R}^n) \times H^m(\mathbb{R}^n)_j^M \\
\downarrow 0 & \downarrow T_j = \text{Primitive Trace Operator} \\
H^{m-j/2}(\partial U) & \to B \to 0 \times H^{m-j/2}(\mathbb{R}^{n-1})^M
\end{align*}
\]

Note \( T_j(Au) = \langle T_j[u \sqrt{a_0}], T_j[p_1^u(u \sqrt{a_1})], \ldots, T_j[p_M^u(u \sqrt{a_M})] \rangle \)
\[
= \langle 0, \partial_n[p_1^u(u \sqrt{a_1})], \ldots, \partial_n[p_M^u(u \sqrt{a_M})] \rangle
\]
and
\[
\partial_n[p_k^u(u \sqrt{a_k})] \in H^{m-j/2}(\mathbb{R}^{n-1}) \text{ for } k = 1, 2, \ldots, M
\]

Since the components of A, B and the primitive trace maps are all continuous, the general trace map is continuous as well (i.e., the composition of continuous mappings is
continuous).

Often we will wish to extend a function defined only on the boundary of a set, into the interior of the set and be able to say that the extended function belongs to some Sobolev space on the large set. Here is a theorem that allows us to do this.

**Theorem 3.5 Extension from the Boundary to the Interior**

Suppose $U$ is a bounded, open and regular set in $\mathbb{R}^n$. Then for any $f \in H^m(\partial U)$ there exists $\tilde{f} \in H^m(U)$ such that $T_0\tilde{f} = f$

**Proof-**

\[
\begin{align*}
H^m(\partial U) &\xrightarrow{A'} \{f_1^\#, \ldots, f_M^\#\} \in H^m(\mathbb{R}^{n-1})^M \\
\downarrow K &\quad H^m(U) < \xrightarrow{B'} \{Kf_1^\#, \ldots, Kf_M^\#\} \in H^m(\mathbb{R}^n)^M
\end{align*}
\]

Here, $K$ denotes the continuous right inverse of the trace operator $T_0$. 