

3. Reaction Diffusion Equations

Consider the following ODE model for population growth

$$u'(t) = a(u(t))u(t), \quad u(0) = u_0$$

where $u(t)$ denotes the population size at time t , and $a(u)$ plays the role of the population dependent growth rate. The model asserts that the population changes at a rate that is proportional to the present population size. Moreover, if we suppose

$$a(u) = \alpha \left(1 - \frac{u(t)}{u_\infty} \right) \quad \text{for constant positive parameters, } \alpha, u_\infty$$

then the population grows when the population is smaller than the "limiting value" u_∞ and the population size decreases if u exceeds this value. Thus the model predicts behavior that is qualitatively consistent with the way we observe populations to behave. This autonomous differential equation has critical points at $u = 0$, and $u = u_\infty$ and it is easy to see that the critical point at zero is unstable while the other is a stable critical point. Then the population $u(t)$ will tend to the limiting value u_∞ as t tends to infinity, regardless of the initial population size. This equation is a special case of the more general autonomous equation, $u'(t) = F(u(t))$.

Now the partial differential equation

$$\partial_t u(x, t) - \nabla^2 u(x, t) = F(u(x, t)) \quad x \in R^n, t > 0, \quad (3.1)$$

can be viewed as an attempt to incorporate the mechanism of diffusion into the population model. We are going to study equations of this form in the case $n = 1$ where the equation can be written more generally as

$$\partial_t u(x, t) - \partial_{xx} b(u(x, t)) = F(u(x, t)), \quad b'(u) > 0.$$

Equations of this form arise in a variety of biological applications and in modelling certain chemical reactions and are referred to as **reaction diffusion equations**.

Example 3.1 Fisher's equation

The reaction diffusion equation with positive constant parameters, D, α, u_∞

$$\partial_t u(x, t) - D \partial_{xx} u(x, t) = \alpha u(x, t) \left(1 - \frac{u(x, t)}{u_\infty} \right), \quad (3.2)$$

is called Fisher's equation and it is usually viewed as a population growth model. The various parameters in the equation have the following dimensions

$$D = \text{diffusivity } (L^2 T^{-1})$$

$$\alpha = \text{growth rate } (T^{-1})$$

$$u_\infty = \text{carrying capacity (numbers of individuals)}$$

Reduction to Dimensionless Form

It is often useful to rewrite the partial differential equation in terms of dimensionless variables. We define

$$\begin{aligned}\tau &= \alpha t && \text{time scaled to the growth rate} \\ z &= x \sqrt{\frac{\alpha}{D}} && \text{distance scaled to diffusion length} \\ v &= u(x, t)/u_{\infty} && \text{population scaled to carrying capacity}\end{aligned}$$

and then the equation becomes

$$\partial_{\tau} v(z, \tau) - \partial_{zz} v(z, \tau) = v(1 - v). \quad (3.3)$$

Existence of a Travelling Wave

Now, in order to investigate the existence of travelling wave solutions, we suppose $v(z, \tau) = V(z - c\tau)$ with $V(s)$ tending to constant values as s tends to plus or minus infinity. Then

$$-cV'(s) - V''(s) = V(s)(1 - V(s)), \quad -\infty < s = z - c\tau < \infty. \quad (3.4)$$

This second order equation reduces to the following autonomous dynamical system

$$\begin{aligned}V'(s) &= W(s) \\ W'(s) &= -cW(s) - V(s)(1 - V(s)).\end{aligned} \quad (3.5)$$

This system has critical points at $(0, 0)$ and $(1, 0)$. Since

$$J(V, W) = \begin{pmatrix} 0 & 1 \\ 2V - 1 & -c \end{pmatrix}$$

we can classify the critical points according to the eigenvalues of this matrix.

$$\text{at } (0, 0) \quad \lambda_{\pm} = \frac{1}{2}(-c \pm \sqrt{c^2 - 4})$$

a stable node if $c > 2$ and a stable focus if $0 < c < 2$

$$\text{at } (1, 0) \quad \lambda_{\pm} = \frac{1}{2}(-c \pm \sqrt{c^2 + 4})$$

a saddle for all values of c .

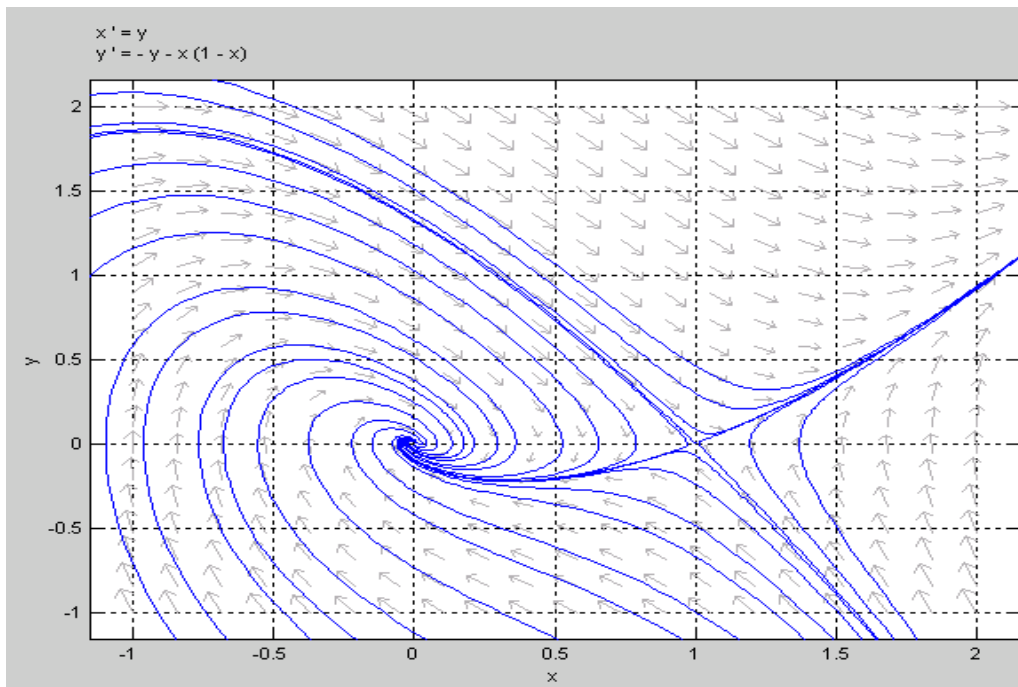
In the case $0 < c < 2$, the origin is a stable focus and the orbits of the system are curves in the (V, W) -plane

$$\left\{ \begin{array}{l} V = V(s), \quad V(-\infty) = V_0 \\ W = W(s), \quad W(-\infty) = W_0 \end{array} \right\} \quad -\infty < s < \infty$$

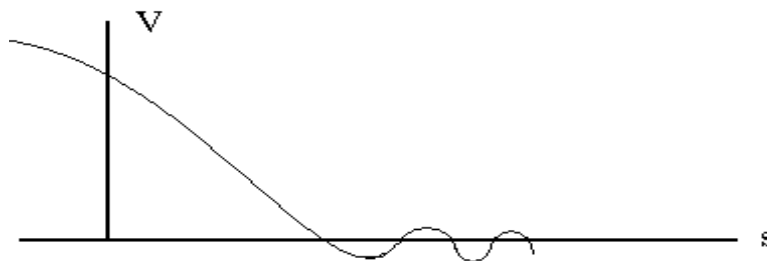
with

$$(V(s), W(s)) \rightarrow (0, 0) \text{ as } s \rightarrow \infty.$$

Since the origin is a focus, the orbits are such that V and W assume both positive and negative values as the curve spirals toward the origin. Negative values for V are not physically meaningful in the population interpretation of $V = u$. Therefore, we conclude that there are no relevant travelling wave solutions for wave speeds between zero and 2.



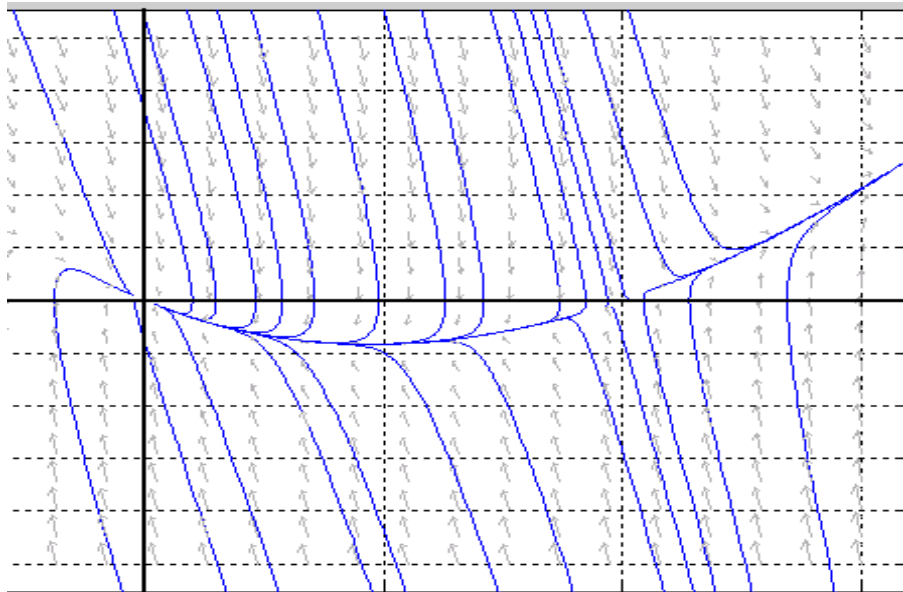
Spiral orbits near the focus



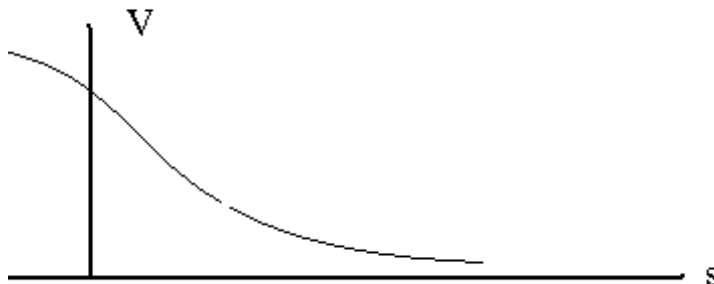
V has both pos and neg values if $0 < c < 2$

In the case $c > 2$ the origin is a stable node and the orbits in the fourth quadrant that are attracted to the origin approach the node with V constantly positive and W constantly negative. Then these are physically relevant orbits. If there exists an orbit with $V_0 = 1, W_0 = 0$ that is attracted to the origin, then this orbit, which is in fact a heteroclinic

orbit joining the two critical points, corresponds to a travelling wave solution to the Fisher's equation. The component $V = V(s)$ of the heteroclinic orbit is a smooth function such that $V'(s) = W(s) < 0$ for all s . In addition, $V(s)$ tends to 1 as s tends to minus infinity and $V(s)$ tends to 0 as s tends to plus infinity so that 0 and 1 are the state values ahead of and behind the wave, respectively. The monotone decreasing function V is the wave form that is propagated at speed $c > 2$.



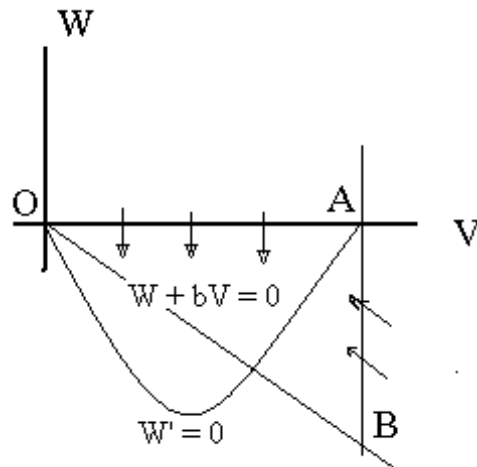
V pos and W neg near stable node



V pos and monotone decreasing if $c > 2$

Proof of Existence of the TW solution

To verify that this heteroclinic orbit actually exists, consider the following picture in the (V, W) - plane.

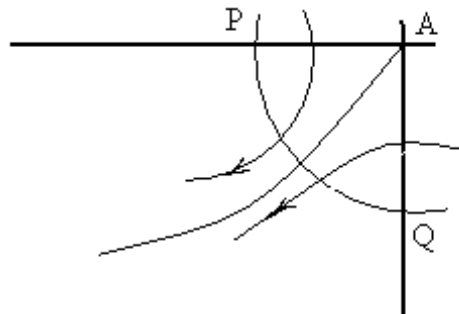


Consider the triangular region, OAB, where the stable node is located at point O, the saddle point is at A, and OB is the line $W + bV = 0$ for some $b > 0$. The parabola $W = \frac{1}{c}V(1 - V)$ is the curve along which $W' = 0$ and $V' = W < 0$.

In addition,

$$\begin{aligned} W' < 0 \text{ and } V' = 0 & \text{ along the line OA,} \\ W' > 0 \text{ and } V' < 0 & \text{ along the line AB.} \end{aligned}$$

Then trajectories that are inside OAB do not leave this region (in the forward direction) through OA or AB. Now consider the region in the neighborhood of point A.



If, by letting s tend to $-\infty$, we trace backward along an orbit which cuts the arc PQ near the point P, we see that such an arc must pass out of OAB above the point A. On the other hand, tracing backward along an orbit which cuts the arc PQ near the point Q, we see that such an arc must pass out of OAB below the point A. It follows that there exists an orbit which passes through the point A as s tends to $-\infty$.

Now consider the line OB, whose equation is $W + bV = 0$ for some positive b . If a trajectory crosses OB from right to left at $s = s_0$ then the expression $W(s) + bV(s)$ decreases from a positive value when $s < s_0$ on the right of OB to a negative value when $s > s_0$ on the left side of OB. Then in order to have a trajectory exit the triangle OAB through OB at

$(V(s_0), W(s_0))$, we must have

$$W'(s_0) + bV'(s_0) < 0 \quad \text{and} \quad W(s_0) + bV(s_0) = 0.$$

On the other hand, the equations of the dynamical system (3.5) imply,

$$\begin{aligned} W'(s) + bV'(s) &= -cW(s) - V(s)(1 - V(s)) + bW(s) \\ &= (b - c)W(s) - V(s)(1 - V(s)). \end{aligned}$$

In particular, along OB

$$\begin{aligned} W'(s_0) + bV'(s_0) &= (b - c)(-bV(s_0)) - V(s_0)(1 - V(s_0)) \\ &= V(s_0)[-b(b - c) - 1 + V(s_0)], \end{aligned}$$

and for $b = c/2 > 0$,

$$W'(s_0) + bV'(s_0) = V(s_0)\left[\frac{c^2}{4} - 1 + V(s_0)\right] > 0 \quad \text{if } c^2 > 4.$$

Then for $c^2 > 4$, no trajectory can exit OAB through OB. Then forward orbits do not exit through any of the sides of the triangle and we say the triangle OAB is a **trapping region** for the dynamical system. It follows that the trajectory that tends to A as s tends to minus infinity must also tend to O as s tends to plus infinity. This is a consequence of the fact that any triangle OA'B' that is similar to OAB is also a trapping region (i.e., it does not permit orbits to exit through any of the sides) and as A', B' tend toward O, the orbits in the triangle are forced toward O. This proves the existence of the orbit joining the saddle at A to the stable node at O. Then the component $V(s)$ satisfies (3.4) and the conditions $V(s) \rightarrow 1$ as $s \rightarrow -\infty$, $V(s) \rightarrow 0$ as $s \rightarrow \infty$. and $v(x, t) = V(x - ct)$ is a TWS for the equation (3.3) as long as $c^2 > 4$. i.e., there is a minimum speed but no maximum.

Estimating the Wave Speed

The existence of arbitrarily high wave speeds is not physically reasonable. In order to resolve this apparent anomaly, consider again the dynamical system (3.5). Note that $W(s) = V'(s)$ is negative along trajectories and therefore at a point $s = s_1$ where $W(s)$ reaches a negative minimum, we have

$$W'(s_1) = 0 \quad \text{or} \quad -cW(s_1) = V(s_1)(1 - V(s_1)) \leq \frac{1}{4},$$

i.e.,

$$\max |V'(s)| \leq |W(s_1)| \leq \frac{1}{4c}.$$

This asserts that the wave speed, c , is inversely proportional to the maximum slope of the wave form $V = V(s)$. That is, a steep wave form travels more slowly than a gradually sloped wave form. Then a bound on the slope of the wave form implies a bound on the speed of the travelling wave. In particular, $\max |\partial_x v(x, 0)| = \max |V'(s)|$ can be estimated from the

data and an upper bound on the speed of the travelling wave solution can be obtained.

Stability of the Travelling Wave

We have seen that the equation (3.3) has a travelling wave solution $v(z, \tau) = V(z - c\tau)$ for $c > 2$. We wish to consider the stability of this solution to see if it is stable under small perturbations. In order to do this, we transform to a moving reference frame, one that is moving with the wave speed, c ; i.e., we let $w(z, \tau) = v(z - c\tau, \tau)$. Then

$$\begin{aligned}\partial_\tau w(z, \tau) &= \partial_\tau v(z - c\tau, \tau) - c \partial_z v(z - c\tau, \tau) \\ &= \partial_\tau w(z, \tau) - c \partial_z w(z, \tau) \\ \partial_\tau w(z, \tau) &= \partial_{zz} w(z, \tau) + c \partial_z w(z, \tau) + w(z, \tau)(1 - w(z, \tau)).\end{aligned}$$

We look in particular for a solution of the form

$$w(z, \tau) = V(z) + \phi(z, \tau)$$

where

$$|\phi(z, \tau)| \ll 1 \quad \text{and} \quad \phi(z, \tau) = 0 \quad \text{for} \quad |z| > L$$

Here L denotes a large positive number. Then, in view of equation (3.4), we have

$$\partial_\tau \phi(z, \tau) = \partial_{zz} \phi(z, \tau) + c \partial_z \phi(z, \tau) + \phi(z, \tau)(1 - 2V(z)) - \phi(z, \tau)^2$$

Since ϕ is assumed to be small, we may neglect the term $\phi(z, \tau)^2$. Then

$$\partial_\tau \phi(z, \tau) = \partial_{zz} \phi(z, \tau) + c \partial_z \phi(z, \tau) + \phi(z, \tau)(1 - 2V(z))$$

and we look for solutions to this equation that have the form $\phi(z, \tau) = \psi(z) e^{-\lambda\tau}$. For a solution of this form we must have

$$\psi''(z) - c\psi'(z) + (\lambda + 1 - 2V(z))\psi(z) = 0, \quad \psi(z) = 0 \quad \text{at} \quad z = \pm L.$$

This is an eigenvalue problem for $\{\lambda_n, \psi_n(z)\}$ where positive eigenvalues imply ϕ decreases with increasing τ (this is a stable solution) and negative eigenvalues imply that ϕ grows with increasing τ (an unstable TW solution). Let $\psi(z) = \theta(z) e^{az}$ and note that for $a = c/2$, we have

$$\theta''(z) + \left[\lambda - \left(2V(z) + \frac{c^2}{4} - 1 \right) \right] \theta(z) = 0, \quad \theta(z) = 0 \quad \text{at} \quad z = \pm L.$$

Note that

$$q(z) := 2V(z) + \frac{c^2}{4} - 1 \geq 2V(z) > 0 \quad \text{if} \quad c > 2.$$

Then

$$\theta(z)\theta''(z) + [\lambda - q(z)]\theta^2(z) = 0$$

and

$$\int_{-L}^L \theta(z)\theta''(z) dz = \theta(z)\theta'(z)|_{z=-L}^{z=L} - \int_{-L}^L \theta'(z)^2 dz = -\int_{-L}^L \theta'(z)^2 dz$$

implies

$$\lambda = \frac{\int_{-L}^L [\theta'(z)^2 + q(z)\theta(z)^2] dz}{\int_{-L}^L \theta(z)^2 dz} > 0.$$

Since the eigenvalues are all positive, the TW solution is stable under small perturbations that vanish outside some finite interval. This is not stability under all possible perturbations but only stability under this special class of perturbations. Nevertheless, it shows that the TW solution has some degree of stability.

Perturbation Approximation for $V(z)$

The TW solution, $V(z)$, satisfies, for $c > 2$,

$$V''(z) + cV'(z) + V(z)(1 - V(z)) = 0, \quad z \in R,$$

$$V(-\infty) = 1, \quad V(\infty) = 0,$$

and since V is translationally invariant we can add the condition that $V(0) = 1/2$ in order to fix the wave. This equation is not exactly solvable for general c and hence we will try to construct an approximate solution. In order to do this, we will make use of the small parameter, $\varepsilon = \frac{1}{c^2}$ which appears in the equation and attempt a so called perturbation expansion in powers of ε .

Let $c = \frac{1}{\sqrt{\varepsilon}}$ (i.e., $\varepsilon = \frac{1}{c^2} < .25$) and write

$$\sqrt{\varepsilon} V''(z) + V'(z) + \sqrt{\varepsilon} V(z)(1 - V(z)) = 0, \quad z \in R,$$

which suggests that for small values of ε (i.e. large c) the dominant term in the ODE is $V'(z)$; i.e., the equation is approximated by $V'(z) = 0$ for large c . The solution $V(z) = \text{constant}$ is valid when z is large negative (there $V(z) \rightarrow 1$) and when z is large and positive (in that region, $V(z) \rightarrow 0$) but there is a large interval "in the middle" where $V(z)$ decreases from 1 to zero and this approximate equation is not valid there. In order to shrink this interval to an interval whose length is of magnitude order one, we introduce a scaled variable

$$s = \frac{z}{c} = \sqrt{\varepsilon} z \quad \text{so} \quad V(z) = V\left(\frac{s}{\sqrt{\varepsilon}}\right) := U(s)$$

Then

$$V'(z) = U'(s)s'(z) = \sqrt{\varepsilon} U'(s), \quad V''(z) = (\sqrt{\varepsilon})^2 U''(s)$$

and

$$0 = \sqrt{\varepsilon} V''(z) + V'(z) + \sqrt{\varepsilon} V(z)(1 - V(z))$$

$$= \sqrt{\varepsilon} \varepsilon U''(s) + \sqrt{\varepsilon} U'(s) + \sqrt{\varepsilon} U(s)(1 - U(s)),$$

Then $U(s)$ solves,

$$\varepsilon U''(s) + U'(s) + U(s)(1 - U(s)) = 0 \quad (3.6)$$

$$U(-\infty) = 1, \quad U(0) = \frac{1}{2}, \quad U(\infty) = 0.$$

Note that the parameter ε appears in the highest order term of the equation, which suggests that the equation obtained by deleting this higher order term is really the dominant part of the problem. To see whether something along these lines is true, we suppose that the solution U can be expanded in a power series in the small parameter ε ,

$$U(s) = U_0(s) + \varepsilon U_1(s) + \varepsilon^2 U_2(s) \dots$$

Substituting this into the equation and side conditions, then equating like powers of ε leads to

$$U_0'(s) = -U_0(1 - U_0), \quad U_0(-\infty) = 1, \quad U_0(0) = \frac{1}{2}, \quad U_0(\infty) = 0, \quad (3.7)$$

$$U_1'(s) = -U_1(1 - 2U_0) - U_0''(s), \quad U_1(\pm\infty) = 0,$$

etc.

This is a series of first order ODE's whose solutions are given by

$$U_0(s) = \frac{1}{1 + e^s}$$

$$U_1(s) = \frac{e^s}{(1 + e^s)^2} \log\left(\frac{4e^s}{(1 + e^s)^2}\right)$$

etc.

Then, finally

$$V(z) = \frac{1}{1 + e^{z/c}} + \frac{1}{c^2} \frac{e^{z/c}}{(1 + e^{z/c})^2} \log\left(\frac{4e^{z/c}}{(1 + e^{z/c})^2}\right) + \dots$$

and this approximation for the wave form $V(z)$ is most accurate for $c \gg 2$ and is least accurate when c is close to 2. In effect, we are saying that the solution to (3.6) consists of the solution to (3.7) plus a series of corrections. Or, saying this another way, the basic behavior of the solution to (3.6) is captured in the solution to (3.7) and the additional terms can be viewed as refinements on this initial approximation. When this solution is compared with numerically constructed solutions, taking just the first term of the expansion gives agreement to within a few percent, even in the worst case when $c = 2$.

We have seen that the equation

$$\partial_t u(x, t) = \partial_{xx} u(x, t) + u(1 - u), \quad x \in R, \quad t > 0,$$

has travelling wave solutions with a minimum wave speed equal to 2. We have not shown that, given an arbitrary initial condition, $u(x, 0)$, the solution will evolve into a travelling wave solution. It has been shown by Kolmogorov that

$$u(x, 0) = \left\{ \begin{array}{ll} 1 & \text{if } x \leq x_1 \\ u_0(x) & \text{if } x_1 \leq x \leq x_2 \\ 0 & \text{if } x \geq x_2 \end{array} \right\}$$

where $u_0(x)$ is continuous, nonnegative and $u_0(x_1) = 1$, $u_0(x_2) = 0$, then the solution $u(x, t)$ will evolve into $u(x, t) = U(x - 2t)$, a travelling wave with the minimal wave speed. For other conditions, the solution may depend on the behavior of $u(x, 0)$ for large $|x|$.

For example, near the leading edge of the wave, $u \approx 0$ so $u^2 \ll u$ and the equation is approximated by the linear equation

$$\partial_t u(x, t) = \partial_{xx} u(x, t) + u.$$

If we consider this equation with $u(x, 0) \approx A e^{-ax}$ as $x \rightarrow \infty$, where $a, A > 0$, then $u(x, t) = A e^{-a(x-ct)}$ solves the equation if

$$ac = a^2 + 1 \quad \text{i.e.,} \quad c = a + \frac{1}{a} \geq 2.$$

If $0 < a < 1$, then

$$e^{-ax} > e^{-x} \text{ and } u(x,0) \approx Ae^{-ax} \text{ as } x \rightarrow \infty \text{ implies } c = a + \frac{1}{a} > 2.$$

On the other hand, if $a > 1$, then

$$e^{-ax} < e^{-x} \text{ and } u(x,0) \approx Ae^{-ax} \text{ as } x \rightarrow \infty \text{ implies } u(x,0) \approx Ae^{-x} \text{ as } x \rightarrow \infty .$$

$$\text{and } u(x,0) \approx Ae^{-x} \text{ as } x \rightarrow \infty \text{ implies } c = 2 .$$

i.e., $u(x,0) \approx Ae^{-ax}$ as $x \rightarrow \infty$ implies $c = a + \frac{1}{a}$, if $0 < a < 1$, $c = 2$ if $a > 1$.

In particular, if $u(x,0)$ has compact support, then $c = 2$. These results have been verified numerically.

The fact that the Fisher equation is invariant under a sign change for x implies that for each TW solution $V(z - c\tau)$ with wave speed c , there is also a TW solution $V(z + c\tau)$ moving in the opposite direction. Given an initial state $v(z,0)$ having compact support, then the initial state will evolve into a pair of travelling wave fronts, one moving to the left and the other to the right. If $v(z,0) < 1$, then the term $v(1 - v) > 0$ and so acts like a source, causing the solution to grow until $v = 1$; i.e., $v(z,\tau) \rightarrow 1$ as $t \rightarrow \infty$ for all x .