7. Systems of Conservation Law Equations Part III

Riemann's Problem.

The initial value problem for a conservation law equation in R_+^2 in which the initial state is piecewise constant, is called ***Riemann's Problem***. For a scalar equation it is the problem

$$\partial_t u(x,t) + \partial_x F(u(x,t)) = 0, \qquad u(x,0) = \left\{ \begin{array}{ll} u_L & if \;\; x < 0 \\ u_R & if \;\; x > 0 \end{array} \right\}.$$

As usual, we suppose $F''(u) > 0$ and we note that the piecewise constant nature of the initial state implies that the characteristics will be straight lines, albeit not all having the same slope. There are three cases to consider,

- $u_L = u_R$: In this case the solution is everywhere equal to the same constant. This is a classical solution.
- $u_L > u_R$: Then $F'(u_L) > F'(u_R)$, which results in a region with overlapping characteristics. In this case the unique weak solution is a shock solution given by

$$u(x,t) = \left\{ \begin{array}{ll} u_L & if \;\; x < s(t) \\ u_R & if \;\; x > s(t) \end{array} \right\} \; t > 0,$$

  where

$$s'(t) = \frac{F'(u_L) - F'(u_R)}{u_L - u_R}, \qquad s(0) = 0.$$

  .
- $u_L < u_R$: Then $F'(u_L) < F'(u_R)$, which results in a wedge shaped region containing no characteristics. In this case the wedge is filled with a rarefaction wave and the unique weak solution is given by

  .

$$u(x,t) = \left\{ \begin{array}{ll} u_L & if \;\; x < F'(u_L)t \\ G\left(\frac{x}{t}\right) & if \;\; F'(u_L)t < x < F'(u_R)t \\ u_R & if \;\; x > F'(u_R)t \end{array} \right\} \; t > 0,$$

  where

$$G(F'(s)) = s, \quad s \in R.$$

To see how this analysis might be extended to systems in R_+^2, consider first, the linear system

$$\partial_t \vec{u} + A\partial_x \vec{u} = \vec{0} \qquad for \; x > 0, \; t > 0,$$

where A denotes a constant n by n matrix with n real, distinct eigenvalues

$$\lambda_1 < \ ... \ < \lambda_k < 0 < \lambda_{k+1} < \ ... \ < \lambda_n.$$

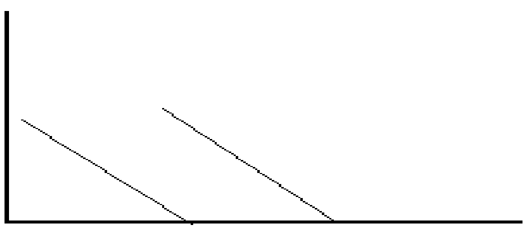
If P denotes the matrix whose columns are the eigenvectors, (in order from 1 to n), of A then $A = P\Lambda P^{-1}$ where $\Lambda = diag\left(\lambda_1, \ ... \ , \lambda_n\right)$. Then

$$\partial_t \vec{v} + \Lambda \partial_x \vec{v} = \vec{0} \qquad for \ x > 0, \ t > 0, \quad where \ \ \vec{v} = P^{-1}\vec{u},$$

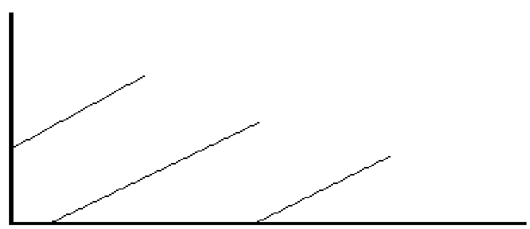
and the system has uncoupled into the n scalar equations

$$\partial_t v_i + \lambda_i \partial_x v_i = 0 \qquad for \quad (x,t) \in Q = \{x > 0, \ t > 0\}, \ 1 \le i \le n.$$

The characteristics for these equations are the straight lines $x - \lambda_i t = x_0$,



$\lambda_i < 0 \quad 1 \le i \le k$



$\lambda_i > 0 \quad k+1 \le i \le n$

For $1 \le i \le k,\ v_i$ is determined in the quarter plane Q by the initial conditions $v_i(x,0) = f_i(x)$. On the other hand, for $k+1 \le i \le n,$ the initial conditions alone are not sufficient to determine v_i in Q. It is also necessary to specify the $n-k$ boundary conditions, $v_i(0,t) = g_i(t),$ for $k+1 \le i \le n.$ Since $\vec{u}$ is composed of linear combinations of $v_i's,$ it follows that $\vec{u}$ is determined throughout Q by the data

$$\left\{ \begin{array}{c} n \quad initial \ conditions \\ n-k \ boundary \ conditions \end{array} \right\} = 2n - k \ conditions.$$

More generally, $\vec{u}$ is determined throughout $Q(t) = \left\{t > 0, \ x \ge s(t)\right\}$ where

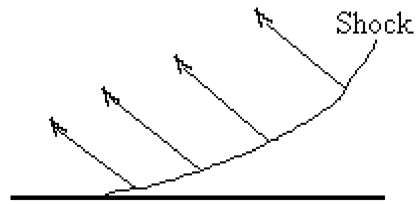
$$\lambda_1 < \ ... \ < \lambda_k < s'(t) < \lambda_{k+1} < \ ... \ < \lambda_n.$$

if we specify n initial conditions and $n-k$ boundary conditions at $x = s(t)$.

Now consider the quasilinear system

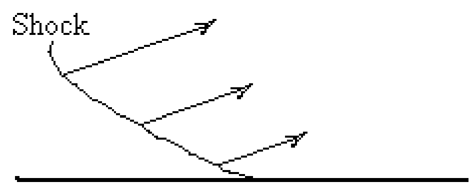
$$\partial_t \vec{u} + A(\vec{u})\, \partial_x \vec{u} = \vec{0} \quad where \quad A_{ij} = \frac{\partial F_i}{\partial u_j}$$

and where A has real distinct eigenvalues, $\lambda_1(\vec{u}) < \ldots < \lambda_n(\vec{u})$. If there is curve $x = s(t)$ in the x-t plane across which $\vec{u}$ jumps from $\vec{u}_L$ *to* $\vec{u}_R$, then by analogy with the linear system, we expect that



$$\lambda_j(\vec{u}_L) < s'(t) < \lambda_{j+1}(\vec{u}_L)$$

j boundary conditions needed at left side of shock



$$\lambda_k(\vec{u}_R) < s'(t) < \lambda_{k+1}(\vec{u}_R)$$

n – k boundary conditions needed at right side of shock

Then a total of $n - k + j$ boundary conditions are needed. We note that the jump condition

$$s'(t)\,(\vec{u}_L - \vec{u}_R) = \vec{F}(\vec{u}_L) - \vec{F}(\vec{u}_R)$$

provides n equations relating $s', \vec{u}_L,$ *and* $\vec{u}_R$. Since $\vec{u}_L - \vec{u}_R \neq \vec{0}$, we can solve for s' in one of the n equations so that we can eliminate s' from the remaining equations, leaving n-1 equations satisfied along the shock curve. In order for the shock solution to be uniquely determined in R^2_+, on both sides of $x = s(t)$, we need $n - 1 = n - k + j$ *or* $j = k - 1$. Then, in order for a shock solution to exist, the quantities $\{s', \vec{u}_L, \vec{u}_R\}$ must satisfy, for some k, $1 \leq k \leq n$,

$$\lambda_k(\vec{u}_R) < s'(t) < \lambda_{k+1}(\vec{u}_R) \quad \text{and} \quad \lambda_{k-1}(\vec{u}_L) < s'(t) < \lambda_k(\vec{u}_L). \tag{7.1}$$

If this condition holds for some k, the resulting solution is called a k-shock. We will consider some examples.

Example 7.1

1. *A scalar equation* $\partial_t u(x,t) + a(u)\,\partial_x u(x,t) = 0.$

Here $\lambda(u) = a(u)$, and since $n = 1$, there is only $\lambda_k(u) = a(u)$ with $k = 1$; there is no

λ_{k+1} *nor* λ_{k-1}. Then (7.1) becomes

$$\lambda_k(u_R) < s'(t), \qquad s'(t) < \lambda_k(u_L) \quad i.e., \quad a(u_R) < s'(t) < a(u_L)$$

which is just the previously derived entropy condition for the admissibility of a shock.

2. *The p-system*

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ p'(u) & 0 \end{bmatrix} \partial_x \begin{bmatrix} u \\ v \end{bmatrix} = \vec{0},$$

$$\vec{u}(x,0) = \begin{Bmatrix} (u_L, v_L) & if \quad x < 0 \\ (u_R, v_R) & if \quad x > 0 \end{Bmatrix}.$$

We assume $p'(u) > 0 \quad and \quad p''(u) > 0$, and in this case,

$$\lambda_1(\vec{u}) = -\sqrt{p'(u)}, \qquad \lambda_2(\vec{u}) = \sqrt{p'(u)},$$

and $n = 2$, so there are two possible cases:

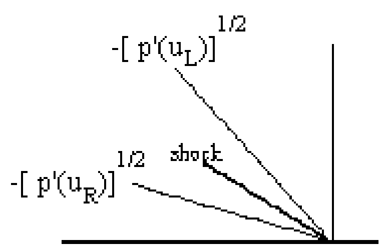
$$k = 1 \qquad \lambda_1(\vec{u}_R) < s'(t) < \lambda_2(\vec{u}_R), \qquad s'(t) < \lambda_1(\vec{u}_L)$$

$$k = 2 \qquad \lambda_2(\vec{u}_R) < s'(t) \qquad \lambda_1(\vec{u}_L) < s'(t) < \lambda_2(\vec{u}_L).$$

Since $\lambda_1(\vec{u}) < 0 < \lambda_2(\vec{u})$, these conditions reduce to

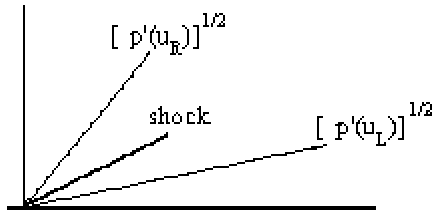
$$k = 1 \qquad \lambda_1(\vec{u}_R) < s'(t) < \lambda_1(\vec{u}_L) < 0,$$

$$k = 2 \qquad 0 < \lambda_2(\vec{u}_R) < s'(t) < \lambda_2(\vec{u}_L),$$



(k = 1) Back Shock

$$k = 1 \qquad -\sqrt{p'(u_R)} < s'(t) < -\sqrt{p'(u_L)} < 0,$$



(k = 2) Front Shock

$$k = 2 \qquad 0 < \sqrt{-p'(u_R)} < s'(t) < \sqrt{-p'(u_L)}.$$

1-shocks and 2-shocks are also referred to as back shocks and front shocks, respectively. To find the back shock, we recall that

$$\vec{F}(\vec{u}) = -\begin{bmatrix} v \\ p(u) \end{bmatrix}$$

hence, the jump conditions become

$$s'(t)(u_L - u_R) = -(v_L - v_R)$$

$$s'(t)(v_L - v_R) = -(p(u_L) - p(u_R)).$$

If either $u_L - u_R = [u]$ *or* $v_L - v_R = [v]$ equals zero, then they both must be zero and there is no back shock. If neither of them is zero, then we can solve for $s'(t)$ in one equation and eliminate it from the other. This leads to

$$v_L - v_R = \pm\sqrt{(u_L - u_R)(p(u_L) - p(u_R))}.$$

Now $-\sqrt{p'(u_R)} < s'(t) < -\sqrt{p'(u_L)} < 0$ implies $0 < p'(u_L) < p'(u_R)$, and since $p''(u) > 0$, it follows that $u_L < u_R; i.e., [u] < 0$. Then substituting $u_L - u_R < 0$, *and* $s'(t) < 0$, into the jump conditions, leads to $v_L - v_R < 0$ *and* $p(u_L) - p(u_R) < 0$. Then we choose the minus sign in the equation above for $v_L - v_R$ to obtain

$$v_R = v_L + \sqrt{(u_L - u_R)(p(u_L) - p(u_R))} := \Sigma_B(u_R), \qquad v_L - v_R < 0.$$

Evidently, for a given $\vec{u}_L$, the set of points $(u_R, v_R) = (u_R, \Sigma_B(u_R))$ comprises the set of all states $\vec{u}_R$ that can be joined to $\vec{u}_L$ by a single back shock. If our value $\vec{u}_R$ falls on this curve then the initial value problem has a back shock solution with shock speed given by

$$s'(t) = -\frac{[v]}{[u]}.$$

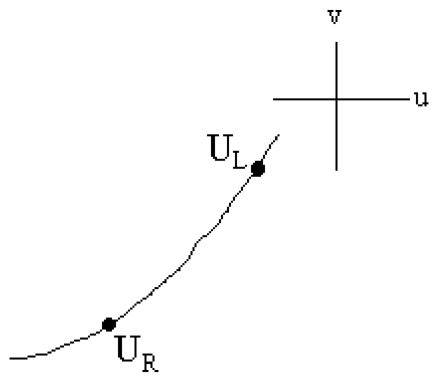
If $\vec{u}_R$ is not on this curve then the back shock is not a physically admissible solution for the initial value problem. Note that

$$\frac{d}{du}\Sigma_B(u) = \frac{-(p(u_L)-p(u))-(u_L-u_R)p'(u)}{2\sqrt{(u_L-u)(p(u_L)-p(u))}}$$

$$= \frac{-(u_L-u)}{2\sqrt{(u_L-u)(p(u_L)-p(u))}}\{p'(u)+\frac{p(u_L)-p(u)}{u_L-u}\}$$

$$> 0 \quad \text{(since } p'(u) > 0\text{)}.$$

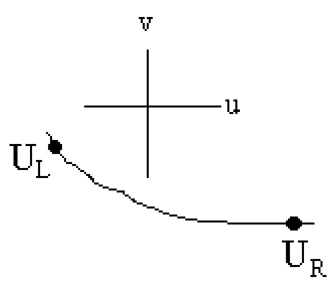
In addition, it can be shown that the second derivative of $\Sigma_B(u)$ is positive and these two facts imply that $v = \Sigma_B(u),$ the curve of admissible back shocks, appears as follows



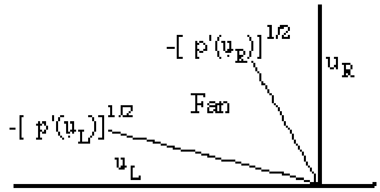
Proceeding similarly for the front shock, where $s' > 0,$ we find
$[u] > 0,\ [p(u)] > 0,\ [v] > 0$ *and*

$$v_R = v_L - \sqrt{(u_L-u_R)(p(u_L)-p(u_R))} := \Sigma_F(u_R), \qquad v_L - v_R > 0.$$

If our value $\vec{u}_R$ falls on the curve $(u_R, \Sigma_F(u_R))$ then the initial value problem has a front shock solution joining $\vec{u}_L$ to $\vec{u}_R$. We can compute the first and second derivatives of $\Sigma_F(u_R)$ and use the signs of these derivatives to conclude that the curve of admissible front shocks has the following appearance

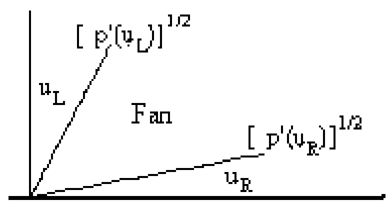


If neither the front shock nor the back shock is admissible, then one of the following two possibilities may occur.



Back Rarefaction Wave

$$k = 1 \qquad \lambda_1(\vec{u}_L) = -\sqrt{-p'(u_L)} < \lambda_1(\vec{u}_R) = -\sqrt{-p'(u_R)} < 0$$



Front Rarefaction Wave

$$k = 2 \qquad 0 < \lambda_2(\vec{u}_L) = \sqrt{-p'(u_L)} < \lambda_2(\vec{u}_R) = \sqrt{-p'(u_R)},$$

In both of these cases, a rarefaction wave solution is of the form, $\vec{u} = \vec{u}\left(\frac{x}{t}\right)$. If we let $\xi = x/t$ then $\vec{u} = \vec{u}(\xi)$ satisfies

$$\partial_t \vec{u}(\xi) = \vec{u}\,'(\xi)\left(\frac{-x}{t^2}\right) \qquad and \qquad \partial_x \vec{F}(\vec{u}(\xi)) = A(\vec{u})\vec{u}'(\xi)\left(\frac{1}{t}\right)$$

hence

$$\vec{u}\,'(\xi)\left(\frac{-x}{t^2}\right) + A(\vec{u})\vec{u}'(\xi)\left(\frac{1}{t}\right) = \left(\frac{1}{t}\right)(-\xi\vec{u}\,'(\xi) + A(\vec{u})\vec{u}'(\xi)) = \vec{0}.$$

If $\vec{u}'(\xi) \neq \vec{0}$ then ξ is an eigenvalue of A with eigenvector $\vec{u}'(\xi)$. In this example, the eigenvalues of A are

$$\lambda_1(\vec{u}) = -\sqrt{p'(u)}, \qquad \lambda_2(\vec{u}) = \sqrt{p'(u)}.$$

Consider the case $\xi = \lambda_1(\vec{u}) = -\sqrt{p'(u)}$. Then we have

$$\begin{bmatrix} -\lambda_1 & -1 \\ -p'(u) & -\lambda_1 \end{bmatrix}\begin{bmatrix} u'(\xi) \\ v'(\xi) \end{bmatrix} = \vec{0},$$

which implies $\lambda_1 u'(\xi) + v'(\xi) = 0$; i.e., $\dfrac{dv}{du} = -\lambda_1(\vec{u}) = \sqrt{p'(u)}$.
Notice that $\xi = \lambda_1$ implies that we have $\lambda_1(\vec{u}_L) < \lambda_1(\vec{u}_R)$ which leads to $p'(u_L) > p'(u_R)$, and this, together with $p''(u) > 0$, implies $u_L > u_R$. Then

$$v_R - v_L = -\int_{u_L}^{u_R} \sqrt{p'(u)}\, du,$$

and we can define

$$v_R = v_L - \int_{u_L}^{u_R} \sqrt{p'(u)}\, du := R_B(u_R), \qquad u_L > u_R,$$

If this function defines a curve $(u_R, R_B(u_R))$ which passes through (u_R, v_R) then $\vec{u}_L$ can be joined to $\vec{u}_R$ by a back rarefaction wave solution. If the curve does not pass through the point then there is no back rarefaction wave solution joining these two states. Note that

$$R_B'(u) = -\sqrt{p'(u)} < 0 \qquad and \qquad R_B''(u) = \frac{-p''(u)}{2\sqrt{p'(u)}} < 0$$

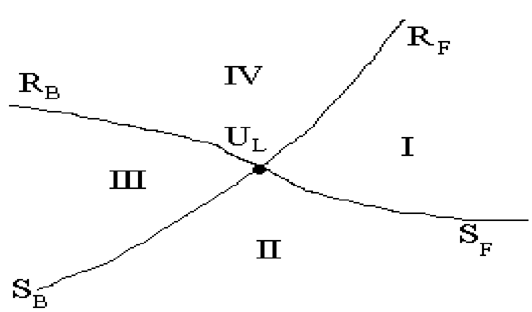
which indicates how the curve $(u, R_B(u))$ must look. In the case $\xi = \lambda_2(\vec{u}) = \sqrt{p'(u)}$, we proceed in a similar fashion to obtain

$$v_R = v_L + \int_{u_L}^{u_R} \sqrt{p'(u)}\, du := R_F(u_R), \qquad u_R > u_L.$$

If the resulting curve $(u_R, R_F(u_R))$ passes through $\vec{u}_R = (u_R, v_R)$ then $\vec{u}_L$ can be joined to $\vec{u}_R$ by front rarefaction wave solution. For the front fan we have

$$R_F'(u) = \sqrt{p'(u)} > 0 \qquad and \qquad R_F''(u) = \frac{p''(u)}{2\sqrt{p'(u)}} > 0.$$

If $\vec{u}_R = (u_R, v_R)$ lies on none of the four curves, $(u_R, \Sigma_B(u_R)), (u_R, \Sigma_F(u_R))$, $(u_R, R_B(u_R))$, $(u_R, R_F(u_R))$ then at each point $\vec{u}_L$ the (u,v) plane can be divided into four regions as indicated in the figure below,



Thus when $\vec{u}_R = (u_R, v_R)$ lies on none of the four curves shown in the figure, it must be that $\vec{u}_R = (u_R, v_R)$ lies in one of the four regions I, II, III, or IV. In this case, we can choose an intermediate state $\vec{U} = (U, V)$ lying on one of the four curves and which can then be joined to (u_R, v_R) by one of the four kinds of curves originating at $\vec{U}$. For example,

i) $\vec{u}_L$ is joined by a back fan to $\vec{U} = (U,V)$ which is joined by a front shock to $\vec{u}_R$.

where $$V = v_L - \int_{u_L}^{U} \sqrt{-p'(u)}\, du := R_B(U), \qquad U > u_L,$$

$$v_R = V - \sqrt{(U - u_R)(p(U) - p(u_R))} := \Sigma_F(u_R), \qquad V > v_R.$$

ii) $\vec{u}_L$ is joined by a back shock to $\vec{U} = (U,V)$ which is joined by a front shock to $\vec{u}_R$

where $$V = v_L + \sqrt{(u_L - U)(p(u_L) - p(U))} := \Sigma_B(U), \qquad v_L < V.$$

$$v_R = V - \sqrt{(U - u_R)(p(U) - p(u_R))} := \Sigma_F(u_R), \qquad V > v_R$$

iii) $\vec{u}_L$ is joined by a back shock to $\vec{U} = (U,V)$ which is joined by a front fan to $\vec{u}_R$

where $$V = v_L + \sqrt{(u_L - U)(p(u_L) - p(U))} := \Sigma_B(U), \qquad v_L < V,$$

$$v_R = V + \int_{U}^{u_R} \sqrt{p'(u)}\, du := R_F(u_R), \qquad u_R > U.$$

iv) $\vec{u}_L$ is joined by a back fan to $\vec{U} = (U,V)$ which is joined by a front fan to $\vec{u}_R$

where $$V = v_L - \int_{u_L}^{U} \sqrt{p'(u)}\, du := R_B(U), \qquad U > u_L,$$

$$v_R = V + \int_{U}^{u_R} \sqrt{p'(u)}\, du := R_F(u_R), \qquad u_R > U.$$

Whether every point in the (u,v) plane is, in fact covered in a unique way by these families of curves so that the process described here can be carried out is something that can be shown (under certain conditions).

Selection Principles

In the section on scalar equations we showed how travelling wave solutions to the equation with artificial viscosity could be used to derive the selection principle for determining when the shock solution is physically relevant. We can try the same approach for the system

$$\begin{aligned} &\partial_t \vec{u}(x,t) + \partial_x \vec{F}(\vec{u}) = \vec{0}. \\ &\vec{u}(x,0) = \left\{ \begin{array}{ccc} \vec{u}_L & if & x < 0 \\ \vec{u}_R & if & x > 0 \end{array} \right\}. \end{aligned} \qquad (7.2)$$

We consider the regularized system

$$\partial_t \vec{u}(x,t) + \partial_x \vec{F}(\vec{u}) = \varepsilon \partial_{xx} \vec{u}(x,t), \qquad (7.3)$$

and look for a travelling wave solution

$$\vec{u}_\varepsilon(x,t) = \vec{V}\left(\frac{x-\sigma t}{\varepsilon}\right) = \vec{V}(s), \quad s = \frac{x-\sigma t}{\varepsilon}.$$

This leads to

$$\vec{V}''(s) = -\sigma \vec{V}'(s) + \nabla\vec{F}\left(\vec{V}(s)\right)\vec{V}'(s) \qquad (7.4)$$

where we suppose that

$$\begin{aligned} &\vec{V}(s) \to \vec{u}_L \quad && as \quad s \to -\infty \\ &\vec{V}(s) \to \vec{u}_R \quad && as \quad s \to +\infty \qquad (7.5) \\ &\vec{V}'(s) \to 0 \quad && as \quad |s| \to \infty. \end{aligned}$$

Then it will follow that as ε tends to zero, this travelling wave solution, $\vec{u}_\varepsilon$, joining $\vec{u}_L$ *to* $\vec{u}_R$, approaches the shock solution for (7.2), namely

$$\vec{u}_\varepsilon(x,t) \to \vec{u}(x,t) = \left\{ \begin{array}{lll} \vec{u}_L & if & x < \sigma t \\ \vec{u}_R & if & x > \sigma t \end{array} \right\} as\ \varepsilon \to 0.$$

To see this, note that (7.4) can be integrated once to get

$$\vec{V}'(s) = \vec{F}\left(\vec{V}\right) - \sigma \vec{V} + \vec{C}_0. \qquad (7.6)$$

The boundary conditions imply that

$$\vec{F}(\vec{u}_L) - \sigma \vec{u}_L + \vec{C}_0 = \vec{F}(\vec{u}_R) - \sigma \vec{u}_R + \vec{C}_0,$$

or

$$\sigma(\vec{u}_L - \vec{u}_R) = \vec{F}(\vec{u}_L) - \vec{F}(\vec{u}_R),$$

and then the ODE (7.6) reduces to

$$\vec{V}'(s) = \vec{F}\left(\vec{V}\right) - \vec{F}(\vec{u}_L) - \sigma(\vec{V} - \vec{u}_L).$$

Now the question is under what conditions can there be a solution to this ODE that joins

$\vec{u}_L$ *to* $\vec{u}_R$. To illustrate how this question might be answered, we consider an example.

Example 7.2

Consider the p-system,

$$\partial_t u(x,t) - \partial_x v(x,t) = 0$$
$$\partial_t v(x,t) - \partial_x p(u) = 0, \qquad where \ \ p'(u) > 0,\ p''(u) > 0.$$

We replace this hyperbolic system by the system,

$$\partial_t u(x,t) - \partial_x v(x,t) = 0$$
$$\partial_t v(x,t) - \partial_x p(u) = \varepsilon \partial_{xx} v(x,t),$$

and we suppose this system has a solution of the form

$$\vec{u}_\varepsilon(x,t) = \vec{W}\left(\frac{x-\sigma t}{\varepsilon}\right) = \vec{W}(\xi) = [w_1(\xi), w_2(\xi)]^\top,$$

where

$$\begin{aligned} &\vec{W}(\xi) \to \vec{u}_L \quad as \ \ \xi \to -\infty, \\ &\vec{W}(\xi) \to \vec{u}_R \quad as \ \ \xi \to +\infty \qquad\qquad (7.7) \\ &\vec{W}'(\xi) \to \vec{0} \quad as \ \ |\xi| \to \infty. \end{aligned}$$

Then

$$\begin{aligned} &-\sigma w_1'(\xi) - w_2'(\xi) = 0, \qquad\qquad (7.8) \\ &-\sigma w_2'(\xi) - (p(w_1(\xi)))' = w_2''(\xi). \end{aligned}$$

After one integration, we obtain

$$\begin{aligned} &-\sigma w_1(\xi) - w_2(\xi) = -\sigma w_1(L) - w_2(L) = -\sigma w_1(R) - w_2(R), \quad (7.9) \\ &w_2'(\xi) = -\sigma(w_2(\xi) - w_2(L)) - (p(w_1(\xi)) - p(w_1(L))) \end{aligned}$$

where

$$\vec{u}_L = (w_1(L), w_2(L)), \qquad \vec{u}_R = (w_1(R), w_2(R)).$$

Then

$$\begin{aligned} &-\sigma(w_1(L) - w_1(R)) = w_2(L) - w_2(R), \\ &-\sigma(w_2(L) - w_2(R)) = p(w_1(L)) - p(w_1(R)) \end{aligned}$$

and

$$\sigma^2 = \frac{p(w_1(L)) - p(w_1(R))}{w_2(L) - w_2(R)} \qquad (7.10)$$

Now it follows from (7.8) and (7.9) that

$$-\sigma w_1'(\xi) = w_2'(\xi) = \sigma^2(w_1(\xi) - w_2(L)) - (p(w_1(\xi)) - p(w_1(L))).$$

That is,

$$w_1'(\xi) = -\sigma(w_1(\xi) - w_2(L)) + \frac{p(w_1(\xi)) - p(w_1(L))}{\sigma} := G(w_1(\xi)).$$

Note that $G(w_1(L)) = 0$, and, as a result of (7.10), $G(w_1(R)) = 0$. In the case that $w_1(L) < w_1(R)$, as ξ goes from minus infinity to plus infinity, $w_1(\xi)$ increases from $w_1(L)$ *to* $w_1(R)$ if and only if $w_1'(\xi) > 0$ *for all* $\xi \in R$. This is the same as requiring $G(w_1) > 0$ *for* $w_1(L) < w_1 < w_1(R)$. That is,

$$-\sigma(w_1(\xi) - w_2(L)) + \frac{p(w_1(\xi)) - p(w_1(L))}{\sigma} > 0,$$

or

$$\frac{p(w_1(\xi)) - p(w_1(L))}{w_1(\xi) - w_2(L)} > \sigma^2 = \frac{p(w_1(L)) - p(w_1(R))}{w_2(L) - w_2(R)} \quad \forall \xi \in R.$$

This last form of the condition for the existence of a TW solution joining $w_1(L)$ *to* $w_1(R)$ (and hence $\vec{u}_L$ *to* $\vec{u}_R$) is somewhat like the condition that in order to have a TW solution for the scalar equation we considered, the wave speeds for the scalar conservation law must be such that the heteroclinic orbit associated with the TW solution travel from u_L *to* u_R.

In the case that $w_1(L) > w_1(R)$, we want $w_1'(\xi) < 0$ *for all* $\xi \in R$ which leads to the opposite of the inequality above.