Problems on Systems of Conservation Laws

1. Consider the system

$$\partial_t u + \partial_x u + \partial_x v = 0$$
$$\partial_t v + \partial_x u + 2\,\partial_x v + \partial_x w = 0$$
$$\partial_t w - \partial_x u + 2\,\partial_x v = 0$$

(a) Write these equations in matrix form and show that the system is strictly hyperbolic
(b) Find the eigenvalues and left eigenvectors for the system
(c) Find the Riemann invariants of the system (there are 3 of them).

2. The equations for 2-dimensional steady state gas dynamics are

$$\partial_x(\rho u) + \partial_y(\rho v) = 0$$
$$u\,\partial_x u + v\,\partial_y u = -\frac{1}{\rho}\partial_x p, \qquad u\,\partial_x v + v\,\partial_y v = -\frac{1}{\rho}\partial_y p$$
$$\rho(u\,\partial_x + v\,\partial_y)\left(\frac{p}{(\gamma-1)\rho} + \tfrac{1}{2}(u^2+v^2)\right) + \partial_x(pu) + \partial_y(pv) = 0$$

(a) Write these equations in matrix form $A\,\partial_x\vec{u} + B\,\partial_y\vec{u} = \vec{0},\ where\ \ \vec{u} = (\rho, u, v, p)$
(b) Show that the system (i.e., $A^{-1}B$) has 4 real but not distinct eigenvalues when $u^2 + v^2 > \gamma p/\rho$
(c) Find any Riemann invariants thay may exist.

3. Consider the system

$$\partial_t u + u\,\partial_x v = u\sqrt{v}$$
$$\partial_t v + v\,\partial_x u = v\sqrt{u}$$

(a) Write these equations in matrix form and show that the system is strictly hyperbolic if $u, v > 0$.
(b) Find the eigenvalues and left eigenvectors for the system
(c) Find the Riemann invariants of the system

4. Consider the system

$$\partial_t u + u\,\partial_x u + 2v\,\partial_x v = 0$$
$$2\,\partial_t v + v\,\partial_x u + 2\,u\,\partial_x v = 0$$

(a) Write these equations in matrix form and show that the system is strictly hyperbolic
(b) Find the eigenvalues and left eigenvectors for the system
(c) Find the Riemann invariants of the system