Existence, Uniqueness and Asymptotic Behavior

In the previous sections of the course we have developed information about particular solutions to various specific equations. In this, the final section, we will prove qualitative results about more general solutions to some nonlinear equations. As is typical for nonlinear problems, none of these results is a result about nonlinear problems in general but each applies to an equation of a specific form but with general ingredients for equations of that form. Since most of the results make use of one or another of a large set of inequalities, we will begin by collecting some of the more useful of these.

1. Inequalities

Lemma 1.1- For $a,b \in R$ *and* $\varepsilon > 0$

i) $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$

ii) $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$

Proof: The first result follows from the observation that for any real numbers a,b

$$\tfrac{1}{2}(a-b)^2 = \tfrac{1}{2}a^2 - ab + \tfrac{1}{2}b^2 \geq 0.$$

The second result follows from

$$\tfrac{1}{2}\left(\sqrt{2\varepsilon}\, a - \frac{b}{\sqrt{2\varepsilon}}\right)^2 = \varepsilon\, a^2 - ab + \frac{1}{4\sqrt{2\varepsilon}}\, b^2 \geq 0.\blacksquare$$

Lemma 1.2 (Young's Inequality) For $1 < p,q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1,\ \textit{and}\ \ a,b > 0,$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof- Note that the function $f(x) = e^x$ is convex, which means that

$$f(\bar{u}) = f\left(\frac{1}{p}x + \frac{1}{q}y\right) \leq \frac{1}{p}f(x) + \frac{1}{q}f(y) = \overline{f(u)} \qquad \forall x,y$$

Then write $$ab = e^{\ln a + \ln b} = e^{\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q} = f(\bar{u})$$

and $$\frac{a^p}{p} + \frac{b^q}{q} = \frac{1}{p}e^{\ln a^p} + \frac{1}{q}e^{\ln b^q} = \overline{f(u)}.\blacksquare$$

Lemma 1.3 (Holder's inequality) For $1 < p,q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1,$ *and* functions u and v, measurable on $U \subset R^n$ such that,

$$\|u\|_p = \left(\int_U |u|^p dx\right)^{1/p}, \qquad \|v\|_q = \left(\int_U |v|^q dx\right)^{1/q},$$

we have

$$\int_U |u\,v|\,dx \leq \|u\|_p \|v\|_q.$$

Proof- WLOG suppose $\|u\|_p = 1, \ \|v\|_q = 1$. Then, using Young's inequality,

$$\int_U |u\,v|\,dx \le \int_U \left(\tfrac{1}{p}|u|^p + \tfrac{1}{q}|v|^q\right) dx = \frac{1}{p} + \frac{1}{q} = \|u\|_p\,\|v\|_q. \blacksquare$$

These inequalities, along with others to be developed later, will be used to prove results of existence, uniqueness and asymptotic behavior of solutions to partial differential equations. These arguments will be carried out in the setting of a linear space of functions. These function spaces will be equipped with a norm and often with an inner product as well.
The function space norms can be defined in various ways, leading then to different linear spaces of function. For functions defined on a bounded open set U in R^n, we may define

for $u \in C(\bar{U})$ $\qquad \|u\|_\infty = \sup_{\bar{U}} |u(x)|$

for $u \in M(U)$ $\qquad \|u\|_{L^p} = \left(\int_U |u|^p dx\right)^{1/p}, \ 1 \le p < \infty.$

$\big(M(U) =$ measurable functions$\big)$.

In particular, when $p = 2$, we will write $\|u\|_{L^2} = \|u\|_0$ for the Hilbert norm associated with the inner product

$$(u,v)_0 = \int_U u\,v\,dx, \qquad \|u\|_0 = (u,u)_0^{1/2}.$$

We define $\quad H^1(U) = \{u \in L^2(U) \mid \partial_{x_j} u \in L^2(U), \ 1 \le j \le n\}$ where $\partial_{x_j} u$ denotes the distributional derivative and

$$\|u\|_1^2 = \|u\|_0^2 + \|\nabla u \bullet \nabla u\|_0$$

is then a Hilbert norm for the inner product,

$$(u,v)_1 = (u,v)_0 + \sum_{j=1}^{n} (\,\partial_{x_j} u, \ \partial_{x_j} v)_0.$$

In the case $n = 1$ the functions in $H^1(U) = H^1[a,b]$ are Lipschitz continuous.

Lemma 1.4 If $u \in H^1[a,b]$ *then*

$(a) \qquad x,y \in [a,b], \quad a \le x < y \le b,$

$$u(y) - u(x) = \int_x^y u'(z)\,dz;$$

$i.e.,$ $\ .u$ is absolutely continuous on $[a,b]$

$(b) \qquad u \in C[a,b]$ (in fact u is Holder continuous with exponent 1/2).

Proof (a) Let $[\alpha,\beta]$ denote an arbitrary subinterval of $[a,b]$. Since we can define $H^1[\alpha,\beta]$ as the completion in the $\|\bullet\|_1$ norm of $C^1[\alpha,\beta]$, it follows that there is a sequence $u_n(x) \in C^1[\alpha,\beta]$ such that u_n converges to u in the $\|\bullet\|_1$ norm. This implies

$$\|u_n - u\|_0 \to 0 \qquad and \qquad \|u_n' - v\|_0 \to 0 \quad \text{as n tends to } \infty$$

Then for each n $\qquad u_n(x) = u_n(\alpha) + \int_\alpha^x u_n'.$

Write this as

$$u_n(\alpha) = u_n(x) - \int_\alpha^x u_n'.$$

Now $u_n(x) - \int_\alpha^x u_n'$ converges in $L^2[\alpha,\beta]$ to the limit $u(x) - \int_\alpha^x v$ and this implies, in turn, that $u_n(\alpha)$ must converge (as a sequence of real numbers) to a limit we will denote by C. Then it follows that

$$u(x) = C + \int_\alpha^x v$$

for almost every x in $[\alpha,\beta]$. By modifying u on a set of measure zero, if needed, we can assume this last equality holds at every point of $[\alpha,\beta]$. Then u is an absolutely continuous function on the interval $[\alpha,\beta]$, and $C = u(\alpha)$; i.e.,

$$u(x) = u(\alpha) + \int_\alpha^x v \qquad \forall x \in [\alpha,\beta].$$

Since $[\alpha,\beta]$ denotes an arbitrary subinterval of $[a,b]$, the point x can be taken as close as we like to either a or to b and we can conclude that $u(x)$ is absolutely continuous on the whole interval $[a,b]$.

(b) For arbitrary $x,y \in [a,b], \quad a \le x < y \le b,$

$$u(y) - u(x) = \int_x^y u'(z)\,dz.$$

Then $$|u(y) - u(x)| \le \int_x^y |u'(z)|\,dz = \int_x^y 1\,|u'(z)|\,dz$$

$$\le \left(\int_x^y 1^2\,dz\right)^{1/2}\left(\int_x^y |u'(z)|^2 dz\right)^{1/2} \le \left(\int_x^y 1^2\,dz\right)^{1/2}\left(\int_a^b |u'(z)|^2 dz\right)^{1/2}$$

$$\le (y-x)^{1/2}\|u'\|_0 \le (y-x)^{1/2}\|u\|_1 \blacksquare$$

Lemma 1.5 (Poincare Inequality) If $u \in H^1[a,b]$ *then*

i) $\|u\|_0 \le 2(b-a)\,\|u'\|_0 \quad if \ u(b) = 0 \ \big(or\ u(a) = 0\big)$

ii) $\|u\|_0 \le \dfrac{b-a}{\pi}\,\|u'\|_0 \quad if \ u(a) = u(b) = 0$

Proof- Suppose $u \in H^1[a,b] \ and \ u(a) = 0.$ Then

$$\int_a^b u(x)^2 dx = \int_a^b 1\,u(x)^2 dx = \int_a^b \tfrac{d}{dx}(x-a)\,u(x)^2 dx$$

$$= (x-a)\,u(x)^2|_{x=a}^{x=b} - \int_a^b 2\,(x-a)\,u(x)\,u'(x)\,dx$$

and

$$\int_a^b |u|^2 dx \le 2(b-a)\int_a^b |u|\,|u'|\,dx \le 2(b-a)\,\|u\|_0\|u'\|_0.$$

Then it follows that $\|u\|_0 \le 2(b-a)\,\|u'\|_0.$ If we have $u(a) = 0$ instead of $u(b) = 0,$ then we use $x-b$ in place of $(x-a)$ in the first line of the proof. If $u(a) = u(b) = 0$ then this estimate can be sharpened as follows. Let $\phi_n(x) = c_n \sin\left(\frac{n\pi(x-a)}{b-a}\right)$ denote the orthonormal family of eigenfunctions for the self adjoint operator $Lu(x) = -u''(x) \ on \ \{u \in H^1[a,b] \mid u(a) = u(b) = 0\}$. Then for any $u \in H^1[a,b]$ we can write

$$u(x) = \sum_n (u,\phi_n)_0 \phi_n(x).$$

In general the convergence of this series is L^2 convergence but if $u \in H^1[a,b]$ and, in addition, $u(a) = u(b) = 0$, then the convergence is uniform and we are entitled to write

$$u'(x) = \sum_n (u,\phi_n)_0 \phi_n'(x).$$

where this series now converges in $L^2[a,b]$. Then

$$\|u'\|_0^2 = \left(\sum_n (u,\phi_n)_0 \phi_n'(x), \sum_m (u,\phi_m)_0 \phi_m'(x)\right)_0 = \sum_n (u,\phi_n)_0 \sum_m (u,\phi_m)_0 (\phi_n',\phi_m')_0.$$

But $\quad (\phi_n',\phi_m')_0 = -(\phi_n'',\phi_m)_0 = \lambda_n(\phi_n,\phi_m)_0$

hence

$$\|u'\|_0^2 = \left(\sum_n (u,\phi_n)_0 \phi_n'(x), \sum_m (u,\phi_m)_0 \phi_m'(x)\right)_0 = \sum_n \lambda_n (u,\phi_n)_0^2$$

$$\geq \lambda_1 \sum_n (u,\phi_n)_0^2 = \lambda_1 \|u\|_0^2 = \frac{\pi}{b-a} \|u\|_0^2 \qquad \blacksquare$$

Corollary A norm is defined on $X = \{ u \in H^1[a,b] \mid u(b) = 0 \}$ *by* $|u|_1 = \|u'\|_0$. and this norm is equivalent to the usual $H^1-norm, \quad \|u\|_1 = \sqrt{(u,u)_0 + (u',u')_0}$. on $H^1[a,b]$.

Note that $|u|_1$ is not a norm on the space $H^1[a,b]$. It is only when the functions vanish at least at one end of the interval that the Poincare inequality holds and $|u|_1$ defines a norm. The reason is that when the function is "starting from zero", $\|u\|_0$ can be estimated in terms of the rate of growth, $\|u'\|_0$.

Lemma 1.6 If $u \in H^1[a,b]$ *then* $u \in C[a,b]$ *and* $\|u\|_\infty \leq C_0 \|u\|_1$.

Proof- Assume that $u \in H^1[a,b]$. We already showed that If $u \in H^1[a,b]$ then $u \in C[a,b]$ and hence there exist points $x_0, x_1 \in [a,b]$ where u assumes its minimum and maximum values on $[a,b]$; $i.e., u(x_0) \leq u(x) \leq u(x_1), \quad a \leq x \leq b$. Since we have assumed $u \in H^1[a,b]$, we can write

$$u(x_1) = u(x_0) + \int_{x_0}^{x_1} u'(x)\,dx.$$

Then $\quad |u(x_1)| \leq |u(x_0)| + \int_{x_0}^{x_1} |u'(x)|\,dx \leq |u(x_0)| + \int_a^b |u'(x)|\,dx$

and $\quad |u(x_1)| = \|u\|_\infty \leq \frac{1}{b-a} \int_a^b |u(x)|\,dx + \int_a^b |u'(x)|\,dx$

$$\leq \frac{1}{b-a} \left(\int_a^b 1^2\,dx\right)^{1/2} \left(\int_a^b |u(x)|^2\,dx\right)^{1/2} + \left(\int_a^b 1^2\,dx\right)^{1/2} \left(\int_a^b |u'(x)|^2\,dx\right)^{1/2}$$

i.e., $\quad \|u\|_\infty \leq C_0 \|u\|_1. \blacksquare$

Example 1.1 A Uniform Bound on the Solution of a Reaction Diffusion Equation
Consider the initial boundary value problem

$$\begin{aligned} \partial_t u(x,t) - \partial_{xx} u(x,t) &= u(x,t)(1-u(x,t)), && 0<x<1,\ t>0, \\ u(x,0) &= f(x), && 0<x<1, \\ \partial_x u(0,t) = 0 &= \partial_x u(1,t), && t>0. \end{aligned} \tag{1.1}$$

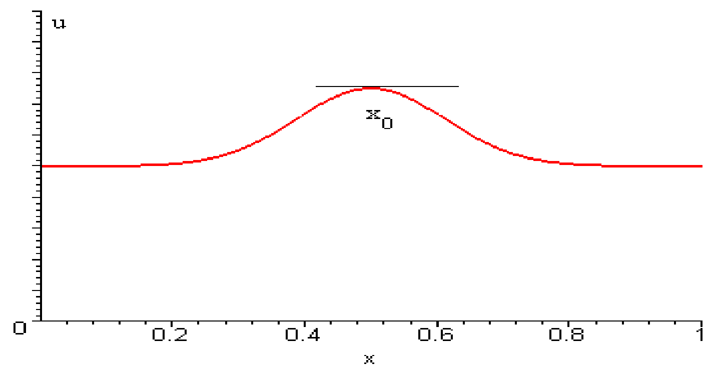
We will show that if this problem has a smooth solution, (i.e., $u, \partial_t u$ and $\partial_{xx} u$ are all continuous on $[0,1]\times[0,\infty)$) and if it is true that

$$\varepsilon \le f(x) \le 1+\varepsilon, \ for \quad 0 \le x \le 1,$$

then it follows that $\quad \varepsilon \le u(x,t) \le 1+\varepsilon, \ for \quad 0 \le x \le 1, \ t \ge 0.$

To prove this result, we will suppose that it is false and show that this assumption leads to a contradiction. Suppose $u(x,t)$ exceeds the value $1+\varepsilon$. Then there must exist some time $t_0 > 0$ and point $x_0 \in [0,1]$ where we have

$$u(x_0,t_0) = 1+\varepsilon, \qquad \partial_{xx} u(x_0,t_0) \le 0, \qquad \partial_t u(x_0,t_0) \ge 0,$$



Then the equation implies

$$\partial_t u(x_0,t_0) = \partial_{xx} u(x_0,t_0) + (1+\varepsilon)(1-(1+\varepsilon)) \le -\varepsilon(1+\varepsilon) < 0$$

and this contradiction implies there can be no such point (x_0,t_0). In the same way, we can show $u(x,t)$ never drops below the value ε. Thus if a smooth solution exists, it must be bounded between ε and $1+\varepsilon$ if this is true initially.

Example 1.2 Asymptotic Behavior of the Solution of a Reaction Diffusion Equation
We continue to consider the initial boundary value problem (1.1) and we will prove some results on the solution behavior as t tends to infinity. First we define

$$F(t) = \int_0^1 \partial_x u(x,t)^2\,dx, \qquad t \ge 0.$$

Note that if we differentiate the equation (1.1) with respect to x, we see that $v(x,t) = \partial_x u(x,t)$ satisfies

$$\begin{aligned} \partial_t v(x,t) - \partial_{xx} v(x,t) &= v(x,t)(1-2u(x,t)), && 0<x<1,\ t>0, \\ v(0,t) = 0 &= v(1,t), && t>0. \end{aligned}$$

Then

$$F'(t) = \int_0^1 2\, v(x,t)\, \partial_t v(x,t)\, dx = \int_0^1 2\, v(x,t)\, [\,\partial_{xx} v + v(1-2u)]\, dx$$

$$= 2\, v\, \partial_x v|_{x=0}^{x=1} - 2\int_0^1 \partial_x v(x,t)^2\, dx + 2\int_0^1 v^2\, (1-2u)\, dx$$

Since u is positive on the interval $[0,1]$, this reduces to

$$F'(t) \le 2\int_0^1 (v^2 - \partial_x v^2) dx = 2\|v\|_0^2 - 2\|\partial_x v\|_0^2.$$

Now using the sharp version of the Poincare inequality, we obtain

$$F'(t) \le 2\|v\|_0^2 - 2\pi^2\, \|v\|_0^2. = -2(\pi^2-1)\, F(t)$$

This implies

$$\frac{d}{dt}\left(e^{2(\pi^2-1)t} F(t)\right) \le 0$$

which leads to $\quad F(t) \le e^{-2(\pi^2-1)t} F(0)$

or $\quad \|\partial_x u(\bullet,t)\|_0^2 \le e^{-2(\pi^2-1)t}\, \|f'\|_0^2.$

This estimate implies that $\partial_x u(\bullet,t)$ converges to zero in $L^2[0,1]$ as t tends to infinity. This does not mean u tends to zero and it does not necessarily mean u tends to a constant but it does mean that as t tends to infinity, $u(x,t) \to u(t)$; *i.e.*, u becomes independent of x.
In order to analyze the asymptotic behavior of the solution, define

$$E(t) = \int_0^1 (u(x,t)-1)^2 dx \quad \textit{for} \quad t \ge 0.$$

Then

$$E'(t) = 2\int_0^1 (u(x,t)-1)\, \partial_t u(x,t)\, dx = 2\int_0^1 (u-1)[\partial_{xx} u + u(1-u)]\, dx$$

$$= 2(u-1)\, \partial_x u\, |_{x=0}^{x=1} - 2\int_0^1 [\partial_x u^2 + u(1-u)^2]\, dx$$

$$\le -2\int_0^1 [u(1-u)^2]\, dx \le -2\varepsilon \int_0^1 (1-u)^2 dx = -2\varepsilon E(t).$$

This implies $\quad E(t) \le e^{-2\varepsilon t} E(0),$

or $\quad \|u(\bullet,t)-1\|_0^2 \le e^{-2\varepsilon t}\, \|f(\bullet)-1\|_0^2 \to 0 \quad as\ t \to \infty.$

The solution tends to the constant 1 as t tends to infinity.

Example 1.3 A More General Uniform Bound on the Solution of a Reaction Diffusion Equation

Consider the following slightly more general reaction diffusion equation,

$$\begin{aligned} \partial_t u(x,t) - \partial_{xx} u(x,t) &= F(u(x,t)), && 0<x<1,\ t>0, \\ u(x,0) &= f(x), && 0<x<1, \\ u(0,t) = 0 &= u(1,t), && t>0. \end{aligned} \tag{1.2}$$

where we suppose $F \in C(R^1)$.
Define

$$E(t) = \int_0^1 [\tfrac{1}{2}\partial_x u(x,t)^2 - G(u(x,t))]\,dx \quad for \quad t \geq 0,$$

where

$$G(u) = \int_0^u F(s)\,ds; \quad i.e. \quad G'(u) = F(u)\ \forall u.$$

Then

$$E'(t) = \int_0^1 [\partial_x u(x,t)\,\partial_{xt}u(x,t) - G'(u(x,t))\,\partial_t u(x,t)]\,dx$$

$$= \int_0^1 [-\partial_{xx}u(x,t)\,\partial_t u(x,t) - F(u(x,t))\,\partial_t u(x,t)]\,dx$$

where we integrated by parts and used the BC's to alter the first term in the integral. Then if $u = u(x,t)$ solves (1.2), this reduces to

$$E'(t) = -\int_0^1 [\,\partial_{xx}u(x,t) + F(u(x,t))\,]\,\partial_t u(x,t)\,dx = -\int_0^1 \partial_t u(x,t)^2\,dx \leq 0 \qquad for \quad t \geq 0.$$

Then $E(t) \leq E(0)\ for \ \ t \geq 0.$ Now suppose F satisfies

$$\limsup_{|u|\to\infty} \frac{F(u)}{u} \leq 0 \qquad (1.3)$$

Note that this condition is satisfied by $F(u) = -u^{2k+1}$ but it is not satisfied for $F(u) = \pm u^{2k}$. We will discuss this condition further when we present the existence proof for this problem. The condition implies that for $\varepsilon > 0,$ there exists a constant C_ε such that

$$G(u) - \varepsilon u^2 = \int_0^u F(s)\,ds - \varepsilon u^2 = \int_0^u \left[\frac{F(s)}{s} - 2\varepsilon\right] s\,ds \leq C_\varepsilon.$$

Then $$E(t) = \int_0^1 [\tfrac{1}{2}\partial_x u(x,t)^2 - G(u(x,t))]\,dx \geq \tfrac{1}{2}\int_0^1 \partial_x u(x,t)^2 dx - \varepsilon\int_0^1 u^2\,dx - C_\varepsilon.$$

In view of the boundary conditions the sharp Poincare inequality implies that for any $\eta,\ 0 < \eta < \frac{1}{2},$

$$\tfrac{1}{2}\int_0^1 \partial_x u(x,t)^2 dx = (\tfrac{1}{2} - \eta)\int_0^1 \partial_x u(x,t)^2 dx + \eta\int_0^1 \partial_x u(x,t)^2 dx$$

$$\geq (\tfrac{1}{2} - \eta)\int_0^1 \partial_x u(x,t)^2 dx + \eta\pi^2\int_0^1 u(x,t)^2 dx$$

and then

$$E(t) \geq (\tfrac{1}{2} - \eta)\int_0^1 \partial_x u(x,t)^2 dx + (\eta\pi^2 - \varepsilon)\int_0^1 u(x,t)^2 dx - C_\varepsilon$$

$$\geq C_\eta\left(\int_0^1 \partial_x u(x,t)^2 dx + \int_0^1 u(x,t)^2 dx\right) - C_\varepsilon = C_\eta\|u\|_1^2 - C_\varepsilon$$

where η is chosen so that $(\frac{1}{2} - \eta) = (\eta\pi^2 - \varepsilon) = C_\eta$. Then we have

$$\|u\|_1^2 \leq \frac{1}{C_\eta}(E(t) + C_\varepsilon) \leq \frac{1}{C_\eta}(E(0) + C_\varepsilon). \qquad (1.4a)$$

But $E(0) = \int_0^1 [\frac{1}{2}f'(x)^2 - G(f(x))]\,dx$ depends only on the initial data so (1.4) may be termed an a-priori estimate for the solution; i.e., it is an estimate on the $H^1 - norm$ of the solution that is based only on the initial data and the function F from the equation. It follows from lemma 1.6 that

$$\|u\|_\infty \le \sqrt{\frac{C_0}{C_\eta}(E(0)+C_\varepsilon)}\,. \qquad (1.4b)$$

We can now use this estimate to prove the existence of a solution to (1.2).

Example 1.4 Existence of a Global Solution for a Reaction Diffusion Equation

We will prove the existence of a global solution for (1.2) under the assumptions

i) $f \in C[0,1]$

ii) $F(u)$ is Lipschitz on R; i.e., *for each* $K > 0,\ \exists M_K > 0,$ *such that*

$$|F(u) - F(v)| \le M_K |u - v| \quad \text{for all } u, v \in R,\ |u|, |v| < K.. \qquad (1.5)$$

iii) $\limsup_{|u|\to\infty} \dfrac{F(u)}{u} \le 0$

We have to recall that the unique solution for the linear problem

$$\begin{aligned} \partial_t v(x,t) - \partial_{xx} v(x,t) &= p(x,t), && 0 < x < 1,\ t > 0,\\ v(x,0) &= f(x), && 0 < x < 1,\\ v(0,t) = 0 &= v(1,t), && t > 0. \end{aligned}$$

can be written as

$$v(x,t) = \int_0^1 G(x,y,t) f(y)\,dy + \int_0^t\int_0^1 G(x,y,t-s)\,p(y,s)\,dyds.$$

Then if u solves (1.2) it follows that

$$u(x,t) = \int_0^1 G(x,y,t) f(y)\,dy + \int_0^t\int_0^1 G(x,y,t-s)\,F(u(y,s))\,dyds \qquad (1.6)$$

Then (1.6) is a nonlinear integral equation with the propert that a solution of (1.2) is a solution of (1.6). If u solves (1.6) then u solves (1.2), at least in some sense, but it may not be a classical solution.

We define a nonlinear mapping

$$\Phi[u] = \int_0^1 G(x,y,t) f(y)\,dy + \int_0^t\int_0^1 G(x,y,t-s)\,F(u(y,s))\,dyds$$

on the linear space $X = C[0,T_0 : C[0,1]]$. Here X is the space of continuous functions of t, $0 \le t \le T_0$, which take their values in the space of continuous functions of x, $0 \le x \le 1$. This space is a complete normed linear space when equipped with the norm

$$\|u\|_\infty = \max\left\{|u(x,t)| : 0 \le x \le 1,\ 0 \le t \le T_0\right\}.$$

Here T_0 denotes a positive time to be specified later. Then for u,v in X, we have

$$\begin{aligned} |\Phi[u] - \Phi[v]| &= \left|\int_0^t\int_0^1 G(x,y,t-s)\big(F(u(y,s)) - F(v(y,s))\big)\,dyds\right| \\ &\le \int_0^t\int_0^1 |G(x,y,t-s)|\,|F(u(y,s)) - F(v(y,s))|\,dyds \end{aligned}$$

hence

$$\|\Phi[u] - \Phi[v]\|_\infty \le \|F(u(y,s)) - F(v(y,s))\|_\infty \int_0^t \int_0^1 |G(x,y,t-s)|\,dyds$$

$$\le M_K \|u - v\|_\infty \int_0^t \int_0^1 |G(x,y,t-s)|\,dyds \le C T_0 M_K \|u - v\|_\infty.$$

Here we used

$$\int_0^t \int_0^1 |G(x,y,t-s)|\,dyds \le C T_0 \qquad 0 \le t \le T_0.$$

Now recall that for F satisfying (1.5iii), the solutions of (1.2) satisfy $\|u\|_\infty \le K$ for some constant K depending only on the initial data. Then the constant M_K in (1.5ii) depends on the initial data as well but the estimate

$$\|F(u(y,s)) - F(v(y,s))\|_\infty \le M_K \|u - v\|_\infty$$

holds uniformly for all solutions u, v of equation (1.2). Then the time T_0 is chosen such that $CM_K T_0 = \alpha < 1$. In this case Φ is a contraction mapping of X into X and there exists then a unique fixed point which must be a solution of (1.6). This fixed point is a local (and possibly weak) solution to the IBVP in the sense that it is valid only on the strip $\{0 \le x \le 1,\ 0 \le t \le T_0\}$. However, since condition (1.5iii) implies that condition (1.5ii) is uniform in u,v, we can make the local solution into a global one. We use the local solution as the initial condition to solve, by the same technique, a second IBVP, one that begins at $t = T_0,\ with\ u = (x, t_0)$. Since the constants K, M_K do not depend on T_0 the solution to the new problem is valid on an inteval of the same width, namely, $[0,1] \times [T_0, 2T_0]$. This process can be repeated over and over, to obtain a solution valid on $[0,1] \times [T_0, nT_0]$ for any integer, n. This extending of a local solution into a global one can be carried out as long as F is uniformly Lipschitz. The combination of (1.5ii) and (1.5iii) result in the needed uniformity. It could also result from an F that is uniformly Lipschitz on R without reference to condition (1.5iii). For example $F(u) = e^{-u^2}$ is is uniformly Lipschitz on R but $F(u) = u^m$ for $m > 0$ is not.

Note that $F(u) = -u^3$ satisfies (1.5iii) but $F(u) = \pm u^2$ does not. Then (1.2) has a global solution for the first choice of F but only a local solution for the second choice. The next example will show why this is the case.

2. Monotonicity Methods

Suppose $u = u(x,t),\ v = v(x,t)$ both satisfy

$$\partial_t u(x,t) - \partial_{xx} u(x,t) = F(u(x,t)), \qquad 0 < x < 1,\ t > 0, \qquad (2.1)$$

$$\partial_x u(0,t) = 0 = \partial_x u(1,t), \qquad t > 0,$$

$$and$$

$$u(x,0) \le v(x,0) \qquad 0 < x < 1. \qquad (2.2)$$

Then $w(x,t) = u(x,t) - v(x,t),$ satisfies

$$\partial_t w(x,t) - \partial_{xx} w(x,t) = F(u) - F(v), \qquad 0 < x < 1,\ t > 0,$$

$$w(x,0) \le 0, \qquad 0 < x < 1,$$

$$\partial_x w(0,t) = 0 = \partial_x w(1,t), \qquad t > 0,$$

But, for $F \in C^1(R)$,

$$F(u)-F(v) = \int_v^u F'(s)\,ds = (u-v)\int_0^1 F'(v+\sigma(u-v))\,d\sigma = p(x,t)\,w(x,t)$$

where

$$p(x,t) = \int_0^1 F'(v(x,t)+\sigma(u(x,t)-v(x,t)))\,d\sigma,\ 0<x<1,\ t>0.$$

Then

$$\partial_t w(x,t) - \partial_{xx} w(x,t) = p(x,t)\,w(x,t)$$

and it follows from the initial and boundary conditions and a maximum principle arguments, that

$$w(x,t) \le 0 \text{ for } 0 \le x \le 1,\ t \ge 0.$$

That is, (2.1) *and* (2.2) imply that $u(x,t) \le v(x,t)$ *for* $0 \le x \le 1,\ t \ge 0$. This result is called a ”comparison principle” and it asserts that for solutions of the equation and conditions in (2.1), the initial ordering (2.2) is preserved as time progresses.

Now suppose $u = u(x,t)$ satisfies

$$\begin{aligned} \partial_t u(x,t) - \partial_{xx}u(x,t) &= F(u(x,t)), && 0<x<1,\ t>0,\\ u(x,0) &= f(x), && 0<x<1, \\ \partial_x u(0,t) = 0 &= \partial_x u(1,t), && t>0, \end{aligned} \qquad (2.3)$$

and $v = v(t)$ solves the ode problem,

$$\begin{aligned} v'(t) &= F(v(t)), \qquad t>0,\\ v(0) &= C. \end{aligned}$$

In particular, let $v_m(t),\ v_M(t)$ denote the solutions in the cases $C = \min_{x\in[0,1]} f(x) = f_m,$ and $C = \max_{x\in[0,1]} f(x) = f_M,$ respectively. Then it follows from the comparison principle that

$$v_m(t) \le u(x,t) \le v_M(t) \qquad for \quad 0 \le x \le 1,\ t \ge 0. \qquad (2.4)$$

Note that $v(t)$ satisfies the derivative boundary conditions in (2.3) since $v(t)$ it is independent of x .

If $F(u) = u^2$ *then* $v(t) = \dfrac{C}{1-Ct}$ *and*

$$\frac{f_m}{1-f_m t} \le u(x,t) \le \frac{f_M}{1-f_M t}$$

If $f_m > 0,$ this result implies that $u(x,t) \to \infty$ *as* $t \to 1/f_m > 0$; i.e., the solution ”blows up” at a finite time. On the other hand, if $f_m \le f_M < 0,$ then $u(x,t) \to 0$ *as* $t \to \infty$. Note that for either of the functions, $F(u) = \pm u^2,$ the solution of the initial boundary value problem could experience blow up at a finite positive time depending on the sign of the initial data.

If $F(u) = -u^3$ then $v(t) = \dfrac{C}{\sqrt{1+2C^2 t}}$ *and*

$$\frac{f_m}{\sqrt{1+2f_m^2 t}} \le u(x,t) \le \frac{f_M}{\sqrt{1+f_M^2 t}}$$

In this case $u(x,t) \to 0 \ as \ t \to \infty,$ whatever the sign of the initial state.

If $F(u) = u(1-u) \ then \ \ v(t) = \dfrac{C}{C+(1-C)e^{-t}} \ \ and$

$$\frac{f_m}{f_m + (1-f_m)e^{-t}} \le u(x,t) \le \frac{f_M}{f_M + (1-f_M)e^{-t}} \qquad .$$

Then $u(x,t) \to 1 \ as \ t \to \infty,$, uniformly in x.

Recall the condition (1.3) from the previous section. Functions $F(u)$ which satisfy this condition behave like $-u^{2k+1}$ for large values of $|u|$. As is illustrated by the example $F(u) = -u^3,$ and as we proved in the previous section, solutions to the initial boundary value problem for $F(u)$ satisfying the condition (1.3) remain bounded for all $t > 0,$ independent of the sign of the initial data. That this condition is sufficient but not necessary for global boundedness is illustrated by the example, $F(u) = u(1-u),$ whose solution is also global in t.

Arguments of this type can be extended to obtain existence results. For convenience, we will write the following IBVP

$$\begin{aligned} \partial_t u(x,t) - \nabla^2 u(x,t) &= F(u(x,t)) \qquad in \ U_T = \{\in U, \ 0 < t < T\} \\ u(x,t) &= g(x,t) \qquad x \in \partial U, \ 0 < t < T, \\ u(x,0) &= u_0(x), \qquad x \in U, \end{aligned}$$

as

$$\begin{aligned} \partial_t u(x,t) - \nabla^2 u(x,t) &= F(u(x,t)) \qquad in \ U_T = \{\in U, \ 0 < t < T\} \\ u(x,t) &= g(x,t) \qquad (x,t) \in \partial U_T \end{aligned}$$

by letting $g(x,0) = u_0(x), \ x \in U$ and letting ∂U_T denote the parabolic boundary of U_T. We will call the function $U_0(x,t)$ an upper solution of the IBVP if

$$\begin{aligned} \partial_t U_0(x,t) - \nabla^2 U_0(x,t) - F(U_0(x,t)) &> 0 \qquad in \ U_T = \{\in U, \ 0 < t < T\} \\ U_0(x,t) &\ge g(x,t) \qquad (x,t) \in \partial U_T \end{aligned}$$

Similarly, $V_0(x,t),$ will be called a lower solution to the IBVP if

$$\begin{aligned} \partial_t V_0(x,t) - \nabla^2 V_0(x,t) - F(V_0(x,t)) &< 0 \qquad in \ U_T = \{\in U, \ 0 < t < T\} \\ V_0(x,t) &\le g(x,t) \qquad (x,t) \in \partial U_T \end{aligned}$$

Assume that upper and lower solutions, $U_0 \ and \ V_0$ are known and suppose further that $F(u)$ is smooth on the interval $J = \{\min V_0 \le u \le \max U_0\}$. Then choose $C > 0$ such that

$$F'(u) + C > 0 \quad for \ all \ u \ in \ J.$$

Now let $U_1 = U_1(x,t)$ denote the solution of the linear problem

$$\partial_t U_1(x,t) - \nabla^2 U_1(x,t) + CU_1(x,t) = F(U_0(x,t)) + CU_0(x,t) \qquad in\ U_T = \{\in U,\ 0 < t < T\}$$
$$U_1(x,t) = g(x,t) \qquad (x,t) \in \partial U_T$$

Since U_1 is completely determined by U_0, we will write, $U_1 = G[U_0]$. Note that

$$\partial_t(U_0 - U_1) - \nabla^2(U_0 - U_1) + C(U_0 - U_1) \geq 0 \qquad in\ U_T = \{\in U,\ 0 < t < T\}$$
$$U_0 - U_1 \geq 0 \qquad (x,t) \in \partial U_T$$

This implies that $U_0 - U_1 > 0\ inside\ U_T;\ i.e.,\ \ U_0 > G[U_0]$. Similarly, if $V_1 = G[V_0]$, we can show $V_1 > V_0$. More generally, we can show

Lemma 2.1 If $u \leq v\ \ then\ \ G[u] \leq G[v]$

This allows us to define sequences, $U_{n+1} = G[U_n]\ \ and\ \ V_{n+1} = G[V_n],\ n = 0, 1, ...$ The lemma implies that these sequences are monotone decreasing and monotone increasing, respectively. If $V_0 \leq U_0$, these sequences converge to a common limit, which is then the solution to the original IBVP. Of course there are numerous technical details to be checked, including whether the sense of the convergence is sufficiently strong to imply that the limit function satisfies the pde in the classical sense.