

## 4. Scalar Conservation Law Equations- part III

We have been introduced to scalar conservation law equations in one space variable at a fairly nontechnical level. In this section we will begin to consider some of the issues that lead us to develop more sophisticated mathematical tools. We begin by reconsidering the so called Lax entropy condition for selecting the physically relevant weak solution to a problem for a scalar conservation law equation.

### Parabolic Regularization

Consider the problem 
$$\partial_t u(x, t) + \partial_x F(u) = 0, \quad u(x, 0) = u_0(x), \quad (4.1)$$

At a point where  $u_L = u(x_-, t) \neq u(x_+, t) = u_R$ , we have the condition that asserts that a shock is the physically relevant solution if  $a(u_L) = F'(u_L) > F'(u_R) = a(u_R)$ . If we suppose  $F''(u) > 0$  for all  $u \in R$  then  $a(u)$  is increasing so  $a(u_L) > a(u_R)$  if and only if  $u_L > u_R$ ; i.e., the wave speed to the left of the discontinuity is greater than the wave speed to the right, the fast wave "crashes into" the slow wave.

An alternative derivation for this selection principle is obtained by considering the perturbed problem,

$$\partial_t u(x, t) + \partial_x F(u) = \varepsilon \partial_{xx} u(x, t), \quad u(x, 0) = u_0(x). \quad (4.2)$$

This problem can be shown to have a unique smooth solution for all  $\varepsilon > 0$ , and for arbitrary initial data in  $L^2(R)$ . The term  $\partial_{xx} u$  has the physical interpretation of viscosity, hence the term  $\varepsilon \partial_{xx} u$  suggests that a small amount of viscosity has been added to the fluid or gas modelled by our original problem. If we can solve the perturbed problem and then let  $\varepsilon \rightarrow 0$ , hopefully the solution will tend to the physically relevant solution of the original zero viscosity problem. The parabolic equation (4.2) is thus a regularized version of equation (4.1) and drawing conclusions about the behavior of the solution to (4.1) by observing behavior in the limit for solutions of (4.2) is referred to as the method of parabolic regularization.

We look for a solution of the form  $u(x, t) = g\left(\frac{x-at}{\varepsilon}\right)$  for  $a \in R$  and  $g \in C^2(R)$  to be determined. Such a solution is called a **travelling wave solution** (TWS) since a solution of this form represents the wave form  $g(\bullet)$  propagating with speed  $a$ . Substituting this solution into (4.2) leads to

$$-\frac{a}{\varepsilon} g'(s) + \frac{1}{\varepsilon} \frac{d}{ds} F(g(s)) = \varepsilon \frac{1}{\varepsilon^2} g''(s) \quad s = \frac{x-at}{\varepsilon};$$

i.e.,

$$g''(s) = \frac{d}{ds} F(g(s)) - a g'(s)$$

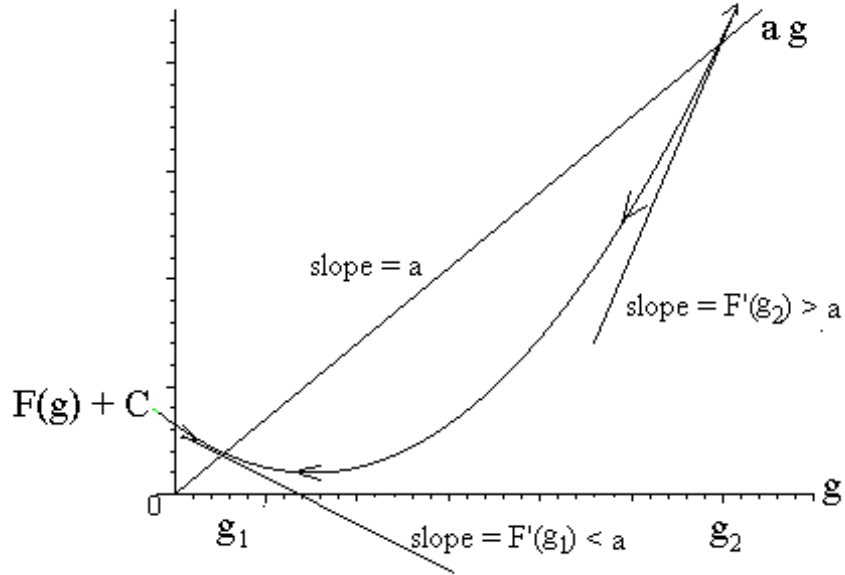
and

$$g'(s) = F(g(s)) - a g(s) + C. \quad (4.3)$$

It follows that  $g'(s)|_{-\infty} = F(g(s))|_{-\infty} - a g(s)|_{-\infty}$  and if we add the constraint that  $g'(s) \rightarrow 0$  as  $|s| \rightarrow \infty$ , then it follows that

$$a = \frac{F(g(\infty)) - F(g(-\infty))}{g(\infty) - g(-\infty)}$$

Note that the speed of the TWS is consistent with a shock speed determined by the R-H jump condition. To examine what happens between  $s = +\infty$  and  $s = -\infty$ , write  $g'(s) = F(g(s)) - ag(s) + C$  and plot the two functions  $f_1(g) = F(g) + C$  and  $f_2(g) = ag$  versus  $g$  on the same axes.

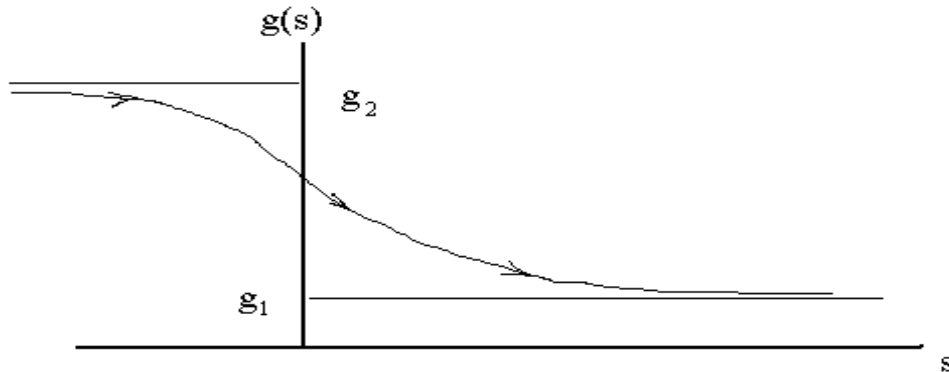


We see from the picture that  $g'(s) = 0$  at  $g = g_1, g_2$  where  $F(g) + a = ag$ . It is also clear from the picture that

i)  $F(g) + a < ag$  for  $g_1 < g < g_2$  hence  $g'(s) < 0$  for  $g_1 < g < g_2$ . (4.4)

ii)  $F(g) + a > ag$  for  $g < g_1$  or  $g > g_2$  hence  $g'(s) > 0$  for  $g < g_1$  or  $g > g_2$ .

These assertions imply that  $g_1$  is a stable fixed point for  $g(s)$  while  $g_2$  is an unstable fixed point for the differential equation for the unknown function,  $g(s)$ . It is also evident from (4.4) that the graph of  $g(s)$  versus  $s$  must look like the following figure,



In the language of dynamical systems, there is a heteroclinic orbit joining the unstable fixed point at  $g = g_2$  to the stable fixed point at  $g = g_1$ . Notice in particular that the stability of the fixed points is determined by the relationship between the slopes of the two functions

$f_1(g) = F(g) + C$  and  $f_2(g) = ag$  at the critical points. Specifically,

$$f_1'(g_2) = F'(g_2) > a = f_2'(g_2) \quad \text{implies} \quad g = g_2 \text{ is unstable}$$

$$f_1'(g_1) = F'(g_1) < a = f_2'(g_1) \quad \text{implies} \quad g = g_1 \text{ is stable.}$$

Then the condition  $F'(g_1) < a < F'(g_2)$  (4.5)

implies the existence of a TWS  $u_\varepsilon(x, t) = g\left(\frac{x-at}{\varepsilon}\right)$  for (4.2).

Now (4.2) tends to (4.1) as  $\varepsilon$  tends to zero, and  $u_\varepsilon(x, 0) = g\left(\frac{x}{\varepsilon}\right)$  tends to the limit  $u_0(x)$  as  $\varepsilon$  tends to zero, where

$$u_0(x) = \begin{cases} g_2 & \text{if } x < 0 \\ g_1 & \text{if } x > 0 \end{cases}. \quad (4.6)$$

i.e.,

$$\begin{array}{ccc} \partial_t u_\varepsilon(x, t) + \partial_x F(u_\varepsilon) = \varepsilon \partial_{xx} u_\varepsilon(x, t), & u_\varepsilon(x, 0) = g\left(\frac{x}{\varepsilon}\right). \\ \downarrow & \downarrow \text{ as } \varepsilon \rightarrow 0 \\ \partial_t u(x, t) + \partial_x F(u) = 0, & u(x, 0) = u_0(x), \end{array}$$

It seems plausible then that as  $\varepsilon \rightarrow 0$ ,  $u_\varepsilon(x, t)$  tends (e.g. in some  $L_p$  - norm) to the solution of (4.1) for  $u_0$  given by (4.6).

In addition, since  $u_\varepsilon(x, t)$  satisfies the entropy condition (4.5) for all values of  $\varepsilon > 0$ , it follows that  $u = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t)$  must also satisfy (4.5). To see that  $u$  is the shock solution, consider the special case,  $F(u) = \frac{1}{2}u^2$ . In this case (4.3) becomes

$$\begin{aligned} g'(s) &= \frac{1}{2}g(s)^2 - ag(s) + C_0 \\ &= \frac{1}{2}[g^2 - 2ag + C_1^2] \\ &= \frac{1}{2}(g - g_1)(g - g_2), \quad g_1 = a - \sqrt{a^2 + C_1^2} < g_2 = a + \sqrt{a^2 + C_1^2}. \end{aligned}$$

It follows now that

$$g(s) = \frac{g_2 + g_1 e^{Ks}}{1 + e^{Ks}}, \quad \text{for } K = \frac{g_2 - g_1}{2}$$

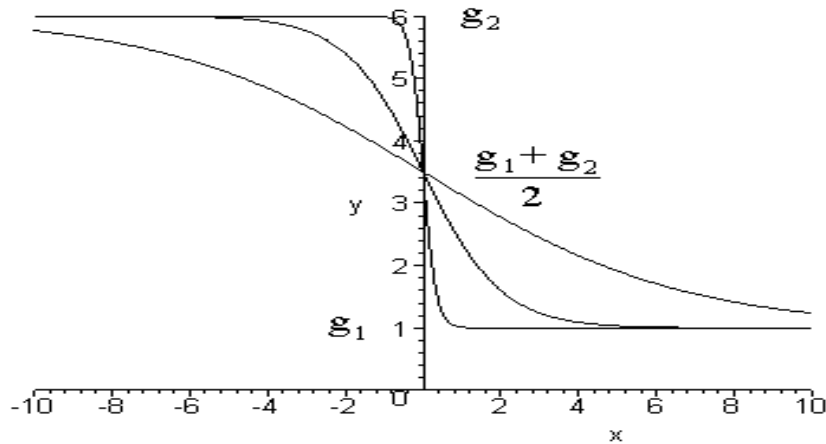
from which we can see :  $g(s) \rightarrow g_2$  as  $s \rightarrow -\infty$ ,

$$g(s) \rightarrow g_1 \quad \text{as } s \rightarrow +\infty,$$

$$g(0) = (g_1 + g_2)/2$$

$$= a \quad (\text{wave speed} = \text{shock speed for Burger's eqn})$$

Note also that plotting  $g\left(\frac{x}{\varepsilon}\right)$  versus  $x$  for various values of  $\varepsilon > 0$ , the travelling wave form  $g$  approaches the step discontinuity as  $\varepsilon \rightarrow 0$ .



graph approaches step as eps tends to 0

### Existence of a Weak Solution

Consider the initial value problem (4.1) where we suppose

- i)  $u_0 \in L_\infty(R)$  (4.7)
- ii)  $F \in C^2(R)$  with  $F''(u) > 0$  for  $|u| \leq \|u_0\|_\infty$

Here  $L_\infty(R)$  is the linear space of functions  $u(x)$  for which there is a finite  $M > 0$  such that  $|u(x)| \leq M$ , almost everywhere. For  $u \in L_\infty(R)$ , the smallest such constant  $M$  is defined to be  $\|u\|_\infty$ .

Under the conditions (4.7) it is possible to prove there exists a weak solution for (4.1). The weak solution  $u = u(x, t)$  has the following properties:

- (a)  $|u(x, t)| \leq \|u_0\|_\infty$  for  $(x, t) \in R_+^2$
- (b) there exists a  $C > 0$ , depending only on  $u_0$  and  $F$  such that for all  $a > 0$ , and all  $(x, t) \in R_+^2$ ,

$$\frac{u(x+a, t) - u(x, t)}{a} < \frac{C}{t}$$

- (c) the solution depends continuously on the data in the sense that if  $u, v$  solve (4.1) for initial data  $u_0, v_0$  respectively,

then for all  $x_1 < x_2$ , and all  $t > 0$

$$\int_{x_1}^{x_2} |u(x, t) - v(x, t)| dx \leq \int_{x_1 - At}^{x_2 + At} |u_0(x) - v_0(x)| dx$$

where  $A = \max \{a(u) : |u| \leq \|u_0\|_\infty\}$ .

The proof of this theorem will not be given (cf shock waves and reaction diffusion equations by Smoller, p266). The proof is constructive, showing that a difference equation approximation to (4.1) is solvable and that the solution to the discrete problem converges in an appropriate sense to the solution of (4.1). Although the details of this process are not given, we can explain some of the implications of the conditions (a), (b), and (c).

Condition (c) asserts that the solution constructed in the proof is unique and depends

continuously on the data. However, it does not assert that there could not be another solution to (4.1) which does not satisfy (c); i.e., although the solution whose existence is implied by the proof has the properties indicated here, there is no guarantee at this point that there is not another solution, obtained by some different algorithm, which does not satisfy (c). Later we will state a theorem that rules out this possibility. Note also that (c) implies that the solution has a finite speed of propagation.

Condition (b) says that if  $u(x+a, t) - u(x, t) < 0$  then the condition holds with any choice of  $C$  (including zero). In this case the difference  $u(x+a, t) - u(x, t)$  need not decay at  $t$  increases. On the other hand, if  $u(x+a, t) - u(x, t) > 0$  then the condition asserts that there is a constant  $C$  for which the estimate holds, and in this case the difference necessarily dies out like  $t^{-1}$ . As we have seen in the examples, for a convex  $F(u)$ , the case  $u(x+a, t) - u(x, t) < 0$  is associated with a shock while the case of a positive difference is associated with a fan or rarefaction wave. Then the condition (b) could be used as a selection principle instead of the previously given entropy condition. Moreover, when the solution is a rarefaction wave, the condition implies that the solution is not only continuous but is "almost differentiable". For example, it follows from (b) that the solution is a function of bounded variation. To see this let  $b$  denote a constant such that  $b > C/t$ , and let  $v(x, t) = u(x, t) - bx$ . Then, for  $a > 0$ , (d) implies

$$v(x+a, t) - v(x, t) = u(x+a, t) - u(x, t) - ba \leq a \left( \frac{C}{t} - b \right) < 0,$$

which is to say,  $v(x, t)$  is nonincreasing as a function of  $x$ . Then  $v(x, t)$  is of bounded variation as a function of  $x$ , and since this is also true of the function  $bx$ , it is also true of  $u(x, t)$ . Now it follows that  $u(x, t)$  has, for each  $t$ , at most a countable number of jump discontinuities as a function of  $x$ .

### Uniqueness of the Weak Solution

Under the conditions (4.7) we can show that, up to a set of measure zero, there is at most one solution for (4.1) which satisfies the condition (b) given above. Again, we cannot give the proof since it is beyond the scope of this course, but we can indicate the idea of the proof. Suppose that  $u$  and  $v$  are two weak solutions for (4.1). Then

$$\iint_{R_+^2} [u \partial_t \psi + F(u) \partial_x \psi] dx dt + \int_R u_0 \psi(x, 0) dx = 0 \quad \forall \psi \in C_c^\infty(R^2)$$

$$\iint_{R_+^2} [v \partial_t \psi + F(v) \partial_x \psi] dx dt + \int_R u_0 \psi(x, 0) dx = 0 \quad \forall \psi \in C_c^\infty(R^2),$$

and

$$\iint_{R_+^2} [(u-v) \partial_t \psi + (F(u) - F(v)) \partial_x \psi] dx dt = 0 \quad \forall \psi \in C_c^\infty(R^2).$$

Now 
$$F(u) - F(v) = \int_0^1 \frac{d}{ds} F(su + (1-s)v) ds = \int_0^1 F'(su + (1-s)v) ds (u-v)$$

hence 
$$F(u(x, t)) - F(v(x, t)) = a(x, t) (u(x, t) - v(x, t))$$

where 
$$a(x, t) = \int_0^1 F'(su(x, t) + (1-s)v(x, t)) ds.$$

Then 
$$\iint_{R_+^2} (u-v) [\partial_t \psi + a(x, t) \partial_x \psi] dx dt = 0 \quad \forall \psi \in C_c^\infty(R^2).$$

Now suppose for arbitrary  $\phi \in C_c^\infty(R^2)$  we can find  $\psi \in C_c^\infty(R^2)$  satisfying

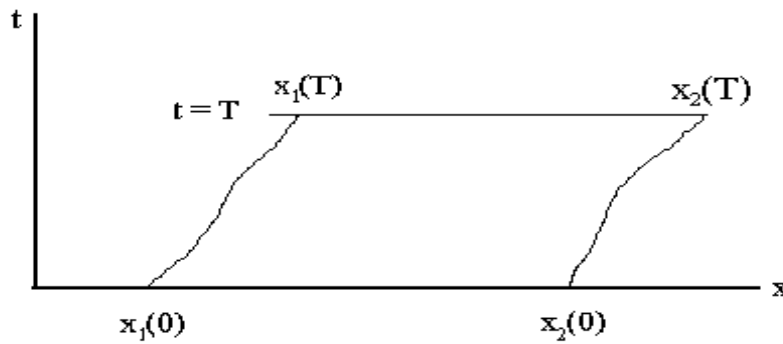
$$\partial_t \psi + a(x,t) \partial_x \psi = \phi \text{ in } R^2. \quad (4.8)$$

Then it would follow that  $\iint_{R^2_+} (u - v) \phi \, dx dt = 0 \quad \forall \phi \in C_c^\infty(R^2)$

which would imply the uniqueness result. It only remains to show that for arbitrary  $\phi \in C_c^\infty(R^2)$  we can find a solution  $\psi$  for (4.8) that belongs to  $C_c^\infty(R^2)$ . This is the difficult part of the proof and it is omitted. It is in this part of the proof that the condition (b) is needed.

### Decay of Entropy Solutions

In addition to existence and uniqueness results, it is possible to prove certain facts about the long term behavior of weak solutions to scalar conservation law equations. These results are the consequence of the entropy condition. Suppose first that  $u(x,t)$  is a smooth solution of (4.1) in the region  $\{(x,t) : x \in I(t), t \in [0, T]\}$  where  $I(t) = (x_1(t), x_2(t))$  and  $x_j(t), j = 1, 2$  are two base characteristics for (4.1).

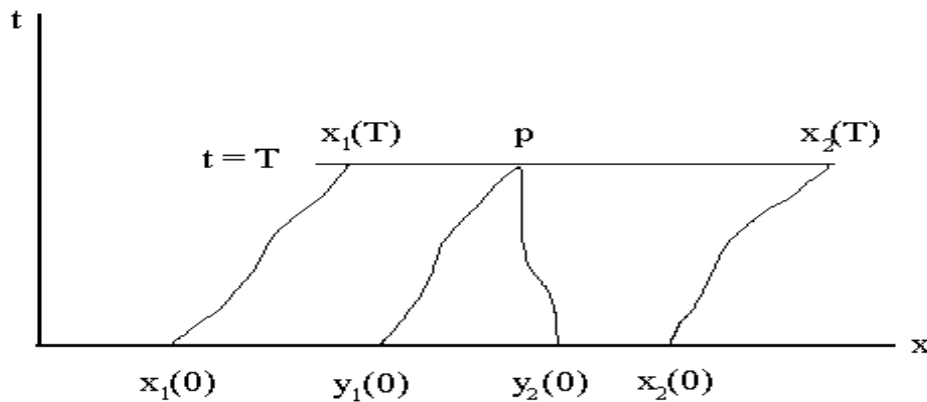


Since  $u$  is constant along characteristics, it follows that the variation of  $u$  on  $I(0)$  in the  $x$ -variable is the same as the variation of  $u$  on  $I(T) = (x_1(T), x_2(T))$  in the  $x$ -variable. That is,

$$\text{var}[u(x, 0), I(0)] = \sup \sum_i |u(x_i, 0) - u(x_{i-1}, 0)|,$$

$$\text{var}[u(x, T), I(T)] = \sup \sum_i |u(x_i, T) - u(x_{i-1}, T)|,$$

where the sup is over all partitions of the interval, and since  $u$  is constant along characteristics, these expressions are equal. Now suppose a shock develops at a point  $p$  of  $I(T)$  as the result of two shocks carrying conflicting information to this point.



Then 
$$\begin{aligned} \text{var}[u(x, T), I(T)] &= \text{var}[u(x, T), (x_1(T), y_1(T))] + \text{var}[u(x, T), (y_2(T), x_2(T))] \\ &= \text{var}[u(x, 0), (x_1(0), y_1(0))] + \text{var}[u(x, 0), (y_2(0), x_2(0))] \\ &< \text{var}[u(x, 0), I(0)]. \end{aligned}$$

This last inequality is the result of the fact that

$$\begin{aligned} \text{var}[u(x, 0), I(0)] &= \text{var}[u(x, 0), (x_1(0), y_1(0))] + \text{var}[u(x, 0), (y_1(0), y_2(0))] \\ &\quad + \text{var}[u(x, 0), (y_2(0), x_2(0))] \end{aligned}$$

and it implies that the variation in  $u(x, t)$  in the  $x$ -variable is a decreasing function of time. This observation can be made quantitatively precise in two different ways.

Suppose  $u = u(x, t)$  solves (4.1). Then,

1. if  $u(x, 0) = u_0(x)$  is periodic then  $u$  tends, uniformly in  $x$  at the rate  $t^{-1}$ , to the mean value of  $u_0(x)$  taken over one period.
2. if  $u(x, 0) = u_0(x)$  has compact support, then

a) 
$$\|u(\cdot, t)\|_\infty \leq \frac{C}{\sqrt{t}}$$

b) 
$$\|u(\cdot, t) - N(\cdot, t)\|_{L_1(\mathbb{R})} \leq \frac{C}{\sqrt{t}}$$

We remark that while 2(a) implies that  $u$  tends to zero in the norm of  $L_\infty(\mathbb{R})$  as  $t \rightarrow \infty$ , this does not imply that  $u$  tends to zero in the norm of  $L_1(\mathbb{R})$ . To see this suppose  $F(0) = 0$  and use the fact that  $u(x, 0) = u_0(x)$  has compact support to write,

$$\frac{d}{dt} \int_{\mathbb{R}} u(x, t) dx = \int_{\mathbb{R}} \partial_t u(x, t) dx = - \int_{\mathbb{R}} \partial_x (F(u(x, t))) dx = -F(u(x, t)) \Big|_{x=-\infty}^{x=\infty} = 0.$$

i.e., since  $u_0(x)$  has compact support and the solution has finite speed of propagation, the derivative is zero. Of course the fact that the area under the graph of  $u(x, t)$  vs  $x$  is constant in time does not mean that the profiles do not change. The implication of 2(b) is that the profile tends asymptotically to a shape called an "N-wave", defined as follows

$$N(x, t) = \left\{ \begin{array}{ll} \frac{1}{d} \left( \frac{x}{t} - \sigma \right) & \text{if } -\sqrt{pdt} < x - \sigma t < \sqrt{qdt} \\ 0 & \text{otherwise} \end{array} \right\}$$

$d = F''(0), \quad \sigma = F'(0), \quad \text{and } p, q \text{ depend on } u_0(x)$

