M 545 Introduction to Linear PDE's

By expressing basic physical principals in mathematical terms, we derive partial differential equations which are then said to model physical systems in the sense that solving the equations allows us to predict the behavior of the physical system under certain conditions. In beginning courses on modeling, such derivations are usually presented in a way that implies that one will always obtain a "good" mathematical problem which only has to be solved in order to proceed with the modeling process. Here we are going to see that a great deal of care must be taken to ensure that the mathematical problem that ensues is "good" in the sense that it has the desirable properties of:

1) **Existence**- a solution to the problem can be found
2) **Uniqueness**- the problem does not have more than one solution
3) **Continuous dependence on the data**- if the input to the problem is slightly changed, the corresponding output (solution) changes only slightly

A problem in partial differential equations that has all of these properties is said to be a ***well posed problem***.
In addition to determining whether our modeling leads to well posed problems, we will consider the question of whether the mathematical properties of the solution to the partial differential equation make physical sense for the corresponding physical system and whether they reveal any unexpected information about the physical system.

Another issue that will be raised is the issue of modifying the setting for the formulation of the mathematical problems to bring them into closer agreement with physical reality. In particular, this refers to weakening the formulation of the problems to accomodate such things as discontinuous or nondifferentiable functions as solutions to partial differential equations. This endeavor requires the introduction of the notion of function spaces and spaces of generalized functions. Associated with these notions will then be the notions of "weaker solutions to partial differential equations". While we will not consider abstract proofs of existence of weak solutions in M545, we will discuss the various weak formulations of PDE's and how such solutions are to be interpreted.

1. Qualitative Properties of Solutions to Linear PDE's

A. Transport Equation

Physical Interpretation
Initial Value Problem
Method of Characteristics

B. Laplace Equation

Physical Interpretation
Mean Value Property and Harmonic Functions

M-m principles
regularity of harmonic functions
Uniqueness for BVP's
M-m principles
energy methods
A Fundamental Solution for the Laplacian
A Solution for Poisson's equation
Green's function for Laplace operator
The Inverse Laplace Operator

C. Heat Equation

Physical Interpretation
M-m principles
on bounded sets
on unbounded sets
Uniqueness for IBVP's
M-m principles
energy methods
A Fundamental Solution for the heat operator
A Solution for Cauchy IVP
Green's formulas for the heat equation
Comparison of solutions for heat and Laplace equations

D. Wave Equation

Physical Interpretation: acoustic waves, E-M waves
D'Alembert sol'n n=1
Wave equation in R^n
Uniqueness for IBVP's
energy methods
Domain of dependence and finite prop speed
Wave-like evolution

E. General Remarks

Classification
Well Posed Problems
Some examples of ill posed problems

2. Elementary Theory for Linear PDE's

A. Function Spaces

Normed Linear spaces: $L_p(U)$

Inner product spaces L_2, ℓ_2
Fourier transform on L_1 –
definition and properties
L_1 –inversion theorem
Fourier transform on L_2
L_2 –inversion theorem
applications, *the* L_2 *derivative*
Sobolev embedding theorem

B. Applications of the Fourier Transform
Laplace's and Poisson's equation
interpretation of equation and boundary conditions
smoothing action of solution operator
Heat Equation
interpretation of equation and initial conditions
infinite speed of propagation
diffusionlike propagation
Wave equation
interpretation of pde and initial conditions
finite speed of propagation
wavelike propagation

C. Orthogonal families and generalized Fourier series in $L^2(U)$
complete orthogonal families
isometry of $L^2(U)$ onto ℓ_2
Sturm-Liouville problems
Hilbert scales: $H_0^s(U)$

D. Applications of Eigenfunction expansions
Laplace's and Poisson's equation
interpretation of equation and boundary conditions
smoothing action of solution operator
Heat Equation
interpretation of equation and initial conditions
smoothing action of solution operator
Wave equation
interpretation of pde and initial conditions
lack of smoothing

Insert **The Mollifier Theorem**

E. Introduction to Distribution Theory

Test functions
Functionals
Distributions
    regular and singular distributions
    differentiation
    convergence
    applications to PDE's
    Hilbert-Sobolev spaces
    distributional Fourier transform
    applications of the distributional transform
Supplement

3. Weak Formulation of Linear PDE's

A. Abstract Hilbert Space Results

Subspaces- $H^1(U)$ *and* $H_0^1(U)$
Projections
Linear Functionals and Bilinear Forms
Lax-Milgram lemma

B. Variational Principles for Physical Systems

Equilibrium systems
    transverse deflection of an elastic membrane
Quadratic functionals and gradients
Variational formulation of BVP's
    stable and natural boundary conditions
Nonsymmetric problems
Additional Variational Problems
    an interface problem
    the biharmonic equation

C. Approximate Solutions for Weak Boundary Value Problems

Approximations subspaces
Approximate problems
Error estimates