By expressing basic physical principals in mathematical terms, we derive partial differential equations which are then said to model physical systems in the sense that solving the equations allows us to predict the behavior of the physical system under certain conditions. In beginning courses on modeling, such derivations are usually presented in a way that implies that one will always obtain a "good" mathematical problem which only has to be solved in order to proceed with the modeling process. Here we are going to see that a great deal of care must be taken to ensure that the mathematical problem that ensues is "good" in the sense that it has the desirable properties of:

1) **Existence** - a solution to the problem can be found  
2) **Uniqueness** - the problem does not have more than one solution  
3) **Continuous dependence on the data** - if the input to the problem is slightly changed, the corresponding output (solution) changes only slightly

A problem in partial differential equations that has all of these properties is said to be a **well posed problem**.

In addition to determining whether our modeling leads to well posed problems, we will consider the question of whether the mathematical properties of the solution to the partial differential equation make physical sense for the corresponding physical system and whether they reveal any unexpected information about the physical system.

Another issue that will be raised is the issue of modifying the setting for the formulation of the mathematical problems to bring them into closer agreement with physical reality. In particular, this refers to weakening the formulation of the problems to accomodate such things as discontinuous or nondifferentiable functions as solutions to partial differential equations. This endeavor requires the introduction of the notion of function spaces and spaces of generalized functions. Associated with these notions will then be the notions of "weaker solutions to partial differential equations". While we will not consider abstract proofs of existence of weak solutions in M545, we will discuss the various weak formulations of PDE’s and how such solutions are to be interpreted.

### 1. Qualitative Properties of Solutions to Linear PDE’s

#### A. Transport Equation
- Physical Interpretation
- Initial Value Problem
- Method of Characteristics

#### B. Laplace Equation
- Physical Interpretation
- Mean Value Property and Harmonic Functions
M-m principles
regularity of harmonic functions
Uniqueness for BVP's
M-m principles
energy methods
A Fundamental Solution for the Laplacian
A Solution for Poisson's equation
Green's function for Laplace operator
The Inverse Laplace Operator

C. Heat Equation
Physical Interpretation
M-m principles
on bounded sets
on unbounded sets
Uniqueness for IBVP's
M-m principles
energy methods
A Fundamental Solution for the heat operator
A Solution for Cauchy IVP
Green's formulas for the heat equation
Comparison of solutions for heat and Laplace equations

D. Wave Equation
Physical Interpretation: acoustic waves, E-M waves
D'Alembert sol'n \( n=1 \)
Wave equation in \( \mathbb{R}^n \)
Uniqueness for IBVP's
energy methods
Domain of dependence and finite prop speed
Wave-like evolution

E. General Remarks
Classification
Well Posed Problems
Some examples of ill posed problems

2. Elementary Theory for Linear PDE's

A. Function Spaces
Normed Linear spaces: \( L_p(U) \)
Inner product spaces $L_2, l_2$
Fourier transform on $L_1$ –
  definition and properties
  $L_1$ – inversion theorem
Fourier transform on $L_2$
  $L_2$ – inversion theorem
  applications, the $L_2$ derivative
  Sobolev embedding theorem

B. Applications of the Fourier Transform
Laplace’s and Poisson’s equation
  interpretation of equation and boundary conditions
  smoothing action of solution operator
Heat Equation
  interpretation of equation and initial conditions
  infinite speed of propagation
  diffusion-like propagation
Wave equation
  interpretation of pde and initial conditions
  finite speed of propagation
  wavelike propagation

C. Orthogonal families and generalized Fourier series in $L^2(U)$
  complete orthogonal families
  isometry of $L^2(U)$ onto $l_2$
  Sturm-Liouville problems
  Hilbert scales: $H^s(U)$

D. Applications of Eigenfunction expansions
Laplace’s and Poisson’s equation
  interpretation of equation and boundary conditions
  smoothing action of solution operator
Heat Equation
  interpretation of equation and initial conditions
  smoothing action of solution operator
Wave equation
  interpretation of pde and initial conditions
  lack of smoothing

Insert The Mollifier Theorem

E. Introduction to Distribution Theory
Test functions
Functionals
Distributions
  regular and singular distributions
differentiation
  convergence
  applications to PDE’s
Hilbert-Sobolev spaces
distributional Fourier transform
  applications of the distributional transform
Supplement

3. Weak Formulation of Linear PDE’s

A. Abstract Hilbert Space Results
  Subspaces- $H^1(U)$ and $H^1_0(U)$
  Projections
  Linear Functionals and Bilinear Forms
  Lax-Milgram lemma

B. Variational Principles for Physical Systems
  Equilibrium systems
    transverse deflection of an elastic membrane
  Quadratic functionals and gradients
  Variational formulation of BVP’s
    stable and natural boundary conditions
  Nonsymmetric problems
  Additional Variational Problems
    an interface problem
    the biharmonic equation

C. Approximate Solutions for Weak Boundary Value Problems
  Approximations subspaces
  Approximate problems
  Error estimates